Graph Theoretical Problems and Related Polytopes: Stable Sets and Perfect Graphs

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[1] This script has been overworked while the author was visiting professor at the Dipartimento di Informatica, Università di L'Aquila

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Chapter 1

Graph Theoretical Problems: A Brief Introduction

1.1 Historical Problems and Basic Definitions

1.1.1 Königsberg's bridges

"Any mathematician who happens to find himself in the East Prussian city of Königsberg (and in the 18th century) will lose no time to follow the great Leonhard Euler's example and inquire about a round trip through the old city that traverses each of the bridges exactly once. (Diestel [26])" Why? This is due to the famous problem posed by Euler:

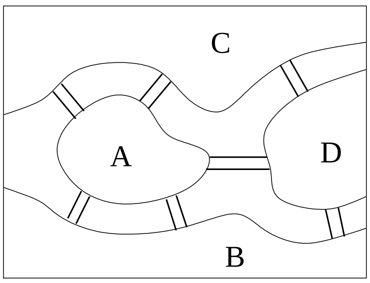
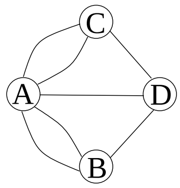


Figure 1.1: The bridges of Königsberg (anno 1736)

Problem 1.1 (Königsberg's bridges problem, Euler 1736 [30]) *Is it possible to start on one shore, to traverse each bridge exactly once, and to turn back to the starting point?*

This was for Euler's Sunday afternoon walk: after having lunch, he wants to walk through the old city and its bridges, and to turn back home for having a coffee or whatever... Euler tried several times to find such a tour - and always failed. He thought more serious about the problem - and this was the starting point of what we now call graph theory:

Problem 1.2 (Reformulation, Euler 1736 [30]) *Construct a graph having all shores and islands of Königsberg as nodes and all bridges as edges. Is there a way starting in one node, passing through all the edges exactly once, and turning back to the start node?*



Remark. The bridges of Königsberg produce a so-called *multigraph* with multiple edges.

Theorem 1.3 (Euler 1736 [30]) *There is no such way traversing the bridges of Königsberg.*

Proof. Such a way exists only if all nodes of the graph are incident to an even number of edges (i.e., whenever we enter a node through an edge, there is always an edge left to leave that node). □

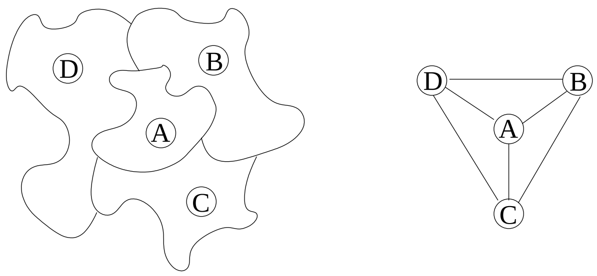
Remark. In fact, Euler characterized in [30] the graphs having a closed walk that passes through every edge precisely once as those graphs where all nodes are incident to an even number of edges.

1.1.2 The Four Color Problem

If any result in graph theory has a claim to be known to the world outside, it is the famous Four Color Theorem. It goes back to the following question:

Problem 1.4 (Four Color Problem, Guthrie 1852) *Is it possible to color every map with four colors s.t. adjacent countries are shown in different colors?*

The graph theoretical reformulation goes as follows: Construct a graph by taking the capitals of the countries as nodes and joining two nodes if the corresponding countries share a border line. The resulting graph is *planar* (i.e. it can be drawn into the plane without crossing lines).



The example shows that four colors may be needed to color maps resp. planar graphs, the question is whether four colors are always *sufficient*.

Generations of mathematicians were busy with the Four Color Problem before it was proved and turned into the Four Color Theorem:

Theorem 1.5 (Four Color Theorem) *Every planar graph can be colored with four colors s.t. no two joined nodes receive the same color.*

- first proof by Appel & Haken [2] in 1977 using a computer (incorrect)

  Very roughly, the proof firstly shows that every plane triangulation must contain at least one of 1482 'unavoidable' configurations. In a second step, a computer is used to show that each of those configurations is 'reducible', i.e., any plane triangulation containing such a configuration can be 4-colored by piecing together 4-colorings of smaller plane triangulations. Taken together, these two steps amount to an inductive proof that all plane triangulations, and hence all planar graphs, can be 4-colored.

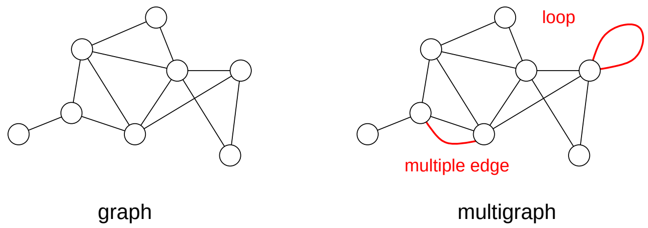
- algorithmic version by Appel & Haken [3] in 1989
  (741 pages long and practically not verifiable due to its length)

- final proof by Robertson, Sanders, Seymour & Thomas [54] in 1997
  (based on the same ideas but more readily, only 40 pages long)

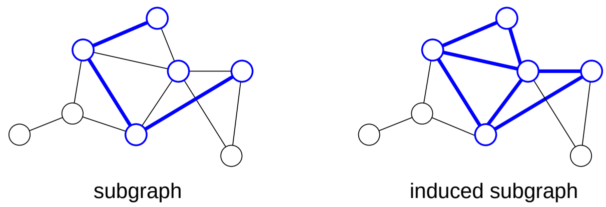
Remark. Appel, Haken & Koch [4] provided in 1977 also a polynomial time algorithm to 4-color planar graphs (based on the incorrect proof). Robertson, Sanders, Seymour & Thomas presented a (correct) coloring algorithm with quadratic running time, see [53] for an outline of the algorithm.

1.1.3 Basic definitions and notations

A **graph** is a pair $G = (V, E)$ of sets satisfying $E \subseteq V^2$, thus the elements of E are 2-element subsets of V. The elements of V are called *nodes* (or vertices, or points), the elements of E *edges* (or lines). The node set of G is referred to as $V(G)$, its edge set as $E(G)$. (No loops or multiple edges are allowed in a graph.)



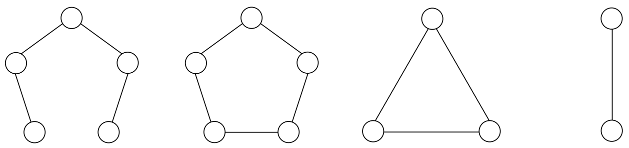
A graph $G' = (V', E')$ with $V' \subseteq V$, $E' \subseteq E$ (where the endnodes of $e \in E$ belong to V') is a **subgraph** of $G = (V, E)$, denoted by $G' \subseteq G$. If $G' \subseteq G$ and G' contains *all* the edges $xy \in E$ with $x, y \in V'$, then G' is an **induced subgraph** of G. We say that V' induces G' and write $G' = G[V']$.



A node v is *incident* with an edge e if $v \in e$. Two nodes $x, y \in V$ are *adjacent* or *neighbors* if xy is an edge. A graph where all nodes are pairwise adjacent is **complete**. The complete graph with n nodes is denoted by K_n.

The set of neighbors of a node x in G is called its *neighborhood* and denoted by $N_G(x)$, or briefly by $N(x)$. The *degree* $d_G(x)$, or briefly $d(x)$, of a node x is the number of nodes in its neighborhood. The number $\delta(G) = \min\{d(v) : v \in V\}$ is the *minimum degree of* G, the number $\Delta(G) = \max\{d(v) : v \in V\}$ its *maximum degree.*

A **path** has distinct nodes $x_1, \ldots, x_k$ and edges $x_i x_{i+1}$ for $1 \leq i < k$. If $P = x_1, \ldots, x_k$ is a path and $k \geq 3$, then the graph $C := P + x_k x_1$ is called a **cycle**. The number of edges in a path or cycle is its *length.* A path or cycle is *even (odd)* if it has even (odd) length. An edge which joins two nodes of a path or cycle but is not itself an edge of the path or cycle, is called a *chord.* A chordless (or induced) path resp. cycle with k nodes is denoted by P_k resp. C_k. If $k \geq 4$ then C_k is called a **hole**.



P_5 C_5 $C_3 = K_3$ $P_2 = K_2$

A non-empty graph G is called *connected* if any two of its nodes are linked by a path in G. A maximal connected (induced) subgraph of G is called a *component*. Note that a component, being connected, is always non-empty.

1.2 Matchings and Stable Sets

1.2.1 Matchings: independent edges

Suppose we are given a graph and are asked to find in it as many independent edges (i.e. edges sharing no end node) as possible. How should we go about this? Will we be able to pair up all nodes of the graph in this way? If not, how can we ensure that this is indeed impossible?

A set M of independent edges in a graph G is called a **matching**. How can we find a *maximum* matching in G, i.e., a matching with as many edges as possible? Let us start by considering an arbitrary matching M in $G = (V, E)$. All nodes in V that are incident (not incident) to an edge in M are called *matched* (*unmatched*). A path in G which contains alternately edges from M and $E - M$ is an M-*alternating path.* (Note: any edge of G can be considered to be an M-alternating path.) An M-alternating path P that starts and ends in an unmatched node of G is called M-*augmenting*, because we can use it to turn M into a larger matching: the symmetric difference of M with $E(P)$ is again a matching $M' = M \triangle E(P)$ of G and satisfies $|M'| = |M| + 1$ (see Figure 1.2).

Theorem 1.6 (Petersen 1891 [52]) *A matching M in a graph G is maximum if and only if there exists no M-augmenting path in G.*

Proof. If M is a maximum matching in $G = (V, E)$, there cannot exist an M-augmenting path P, since otherwise the symmetric difference $M \triangle E(P)$

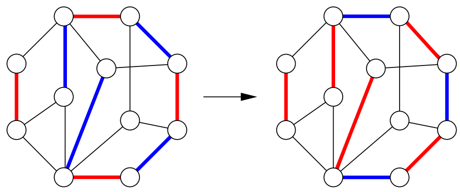


Figure 1.2: Augmenting a matching.

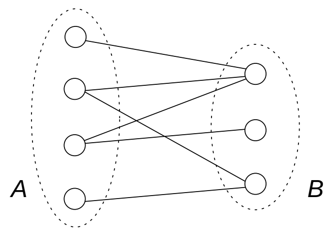
would be a larger matching. Conversely, if M' is a matching larger than M, consider the components of the graph $G' = (V, M \cup M')$. Then G' has maximum degree two, hence each component of G' is either a chordless path (possibly of length 0) or a chordless cycle. The assumption $|M'| > |M|$ implies that at least one of these components should contain more edges in M' than in M. Such a component forms an M-augmenting path, a contradiction. □

So in any graph, if we have an algorithm finding an M-augmenting path for any matching M, then we can find a maximum matching: we iteratively find matchings $M_0, M_1, \ldots$, with $|M_i| = i$, until we have a matching M_k s.t. no M_k-augmenting path exits anymore. (This was observed by Petersen in [52], too.)

One special kind of matching problems is of the following type:

Problem 1.7 (Assignment Problem) *Given a set A of jobs, a set B of candidates, and a set E of pairs $\{a, b\}$ telling that candidate $b \in B$ is suitable for job $a \in A$. Find an assignment of jobs to candidates s.t. as many jobs as possible are matched.*

This kind of problems can be interpreted as a matching problem in bipartite graphs. A graph $G = (A \cup B, E)$ is called **bipartite** if its node set can be partitioned into two disjoint sets A and B s.t. every edge in E has one endnode in A, the other endnode in B:



The above assignment problem is equivalent to finding a maximum matching in a bipartite graph. A classical min-max relation due to König [42] characterizes the size of a maximum matching in a bipartite graph. To this end, call a set C of nodes of a graph G a **node cover** if each edge of G intersects C. Define the **matching number** $\nu(G)$ as the maximum size of a matching in G and the **node cover number** $\tau(G)$ as the minimum size of a node cover in G. It is easy to see that, for any graph G,

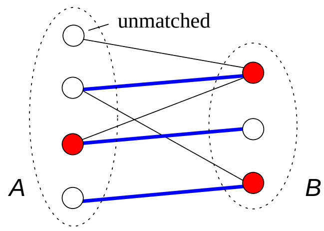
$$\nu(G) \leq \tau(G) \tag{1.1}$$

holds since no two edges in any matching can be covered by the same node. All odd cycles have strict inequality in (1.1) by $\nu(C_{2k+1}) = k$ but $\tau(C_{2k+1}) = k+1$. However, König showed that equality holds in (1.1) whenever the graph G is bipartite.

Prove as exercise: A graph is bipartite iff it contains no odd cycle.

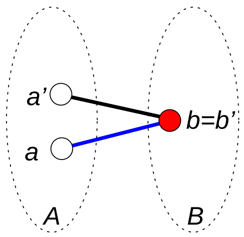
Theorem 1.8 (Matching Theorem, König 1931 [42]) *Any bipartite graph G satisfies $\nu(G) = \tau(G)$. That is, the size of a maximum matching in a bipartite graph equals the minimum size of a node cover.*

Proof. By (1.1) it suffices to show $\nu(G) \geq \tau(G)$. We may assume that $G = (A \cup B, E)$ has at least one edge. Consider a maximum matching M in G. We construct a node set U as follows: from every edge $ab \in M$ let us choose one of its endnodes for U, namely, $b \in B$ if some M-alternating path starting in an unmatched node in A ends in b, and $a \in A$ otherwise. The figure below shows an example (the edges in M are drawn in blue, the nodes in U are red-filled).

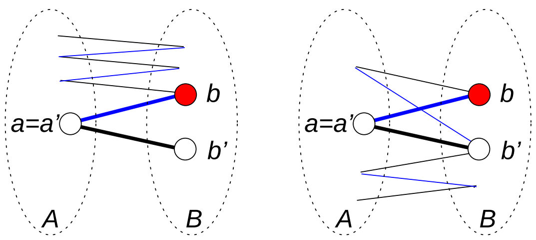


We shall establish that the set U of these $|M|$ nodes covers all edges of G. Let $a'b' \in E$ be an edge, we show that either a' or b' lies in U. If $a'b' \in M$ this holds by the construction of U. Hence assume $a'b' \notin M$. Since M is a maximum matching, it contains an edge ab with $a = a'$ or $b = b'$.

If a' is unmatched then $a \neq a'$ and $b = b'$ follows (since $a'b' \notin M$ and $ab \in M$ share a node) and $a'b'$ is an M-alternating path starting in an unmatched node in A. Thus the endnode of ab chosen for U was $b = b'$.



If a' is matched let $a = a'$. In case $a = a'$ does not belong to U, then $b \in U$ follows and some M-alternating path P starting in an unmatched node a'' in A ends in b. (Note that P cannot contain a since the path had to enter a through an edge in M different from ab.) But then there is also an M-alternating path P' starting in a'' and ending in b': $P' = P \cup \{ba', a'b'\}$ if $b' \notin P$ or $P' = P[a'', b']$ if $b' \in P$.



By the maximality of M, however, P' cannot be M-augmenting. Hence b' must be matched and was chosen for U from the edge in M containing it. □

There are several polynomial time algorithms to find a maximum matching in a bipartite graph. The running time of such algorithms has been improved over the years, e.g., from $O(|V|\ |E|)$ by Kuhn [43] in 1955, to $O(\sqrt{|V|}|E|)$ by Hopcroft & Karp [39] in 1973, and finally to $O(\sqrt{\nu(G)}|E|)$.

1.2.2 Stable sets: independent nodes

Analogously to matchings, we can ask for finding in a graph as many independent nodes as possible. How should we go about this? The problem looks, at the first moment, similar to the matching problem. However, finding independent node sets has turned out to be much harder.

A **stable set** is a set of pairwise non-adjacent nodes. Define the **stability number** $\alpha(G)$ as the cardinality of a *maximum* stable set in G, i.e., of a stable set with as many nodes as possible.

The *stable set problem* is to decide, for a given graph G and a natural number k, whether $\alpha(G) \geq k$ holds. By reducing the satisfiability problem to the stable set problem, Karp showed:

Theorem 1.9 (Karp 1972 [40]) *Determining the stability number of an arbitrary graph is NP-complete.*

1.3 Cliques and Colorings

1.3.1 Coloring the nodes of a graph

The starting point was the question how many colors do we need to color the countries of a map s.t. adjacent countries are colored differently (Four Color Problem). This problem turned out to be equivalent to the (node) coloring problem in a special class of graphs, the planar graphs.

A **coloring** of a graph $G = (V, E)$ is a mapping $c : V \to \mathcal{C}$ s.t. adjacent nodes get different colors, i.e., $c(x) \neq c(y)$ if $xy \in E$. The elements of the set $\mathcal{C}$ are the available colors. If $|\mathcal{C}| = k$, then $c : V \to \{1, \ldots, k\}$ is a *k-coloring*. The smallest k s.t. G admits a k-coloring is the **chromatic number** $\chi(G)$. A graph G is k-chromatic if $\chi(G) = k$ and k-colorable if $\chi(G) \leq k$.

Remark. A k-coloring of G is nothing but a partition of its node set into k stable sets, called *color classes*. The 2-colorable graphs, for example, are the bipartite graphs; the 2-chromatic graphs are the bipartite graphs having at least one edge.

By reducing the stable set problem to the coloring problem, Karp showed:

Theorem 1.10 (Karp 1972 [40]) *Determining the chromatic number of an arbitrary graph is NP-complete.*

How can we, nevertheless, determine the chromatic number of a given graph? How can we *find* a coloring with as few colors as possible? How does the chromatic number relate to other graph parameters (in order to obtain upper and lower bounds on it)?

1.3.2 Greedy colorings and Brook's theorem

One obvious way to color a graph G with not too many colors is the following *greedy algorithm*: starting from a fixed ordering $v_1, \dots, v_n$ of the nodes of G, we consider the nodes in turn and color each node v_i with the first available color (e.g. with the smallest integer not already used to color any neighbor of v_i among $v_1, \dots, v_{i-1}$).

Figure 1.3: Greedy colorings using the lexicographic order on the nodes

Remark. For any graph G, there is an ordering of the nodes of G s.t. the greedy algorithm produces a $\chi(G)$-coloring *(Proof as exercise)*.

The greedy coloring implies a trivial upper bound on the chromatic number

$$\chi(G) \leq \Delta(G) + 1 \tag{1.2}$$

since we never use more than $\Delta(G) + 1$ colors, even for unfavorable choices of the ordering: in the worst case, a node of maximum degree comes in the order after all its neighbors and all of them received different colors.

If G is complete or an odd hole, then this is even best possible:

- $\Delta(K_n) = n - 1$ and $\chi(K_n) = n$
- $\Delta(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$

As we have seen, every graph G satisfies (1.2), with equality for complete graphs and odd holes. In all other cases, this bound can be slightly improved:

Theorem 1.11 (Brooks' Theorem, Brooks 1941 [9]) *For a connected graph G, one has $\chi(G) \leq \Delta(G)$, provided G is neither complete nor an odd hole.*

However, the gap between $\chi(G)$ and $\Delta(G)$ can be arbitrarily large. A bipartite graph $G = (A \cup B, E)$ is **complete** if E contains *all* possible edges between A and B. We denote the complete bipartite graph with $|A| = a$, $|B| = b$ by $K_{a,b}$. Obviously, we have:

Lemma 1.12 *The complete bipartite graphs $K_{1,k}$ satisfy $\chi(K_{1,k}) = 2$ but $\Delta(K_{1,k}) = k$.*

An immediate consequence is therefore:

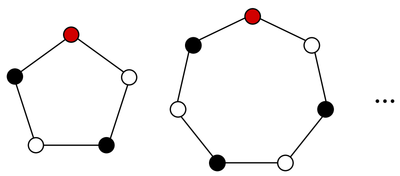
Corollary 1.13 *Let G be connected and neither complete nor an odd hole. The upper bound $\Delta(G)$ for $\chi(G)$ can be arbitrarily bad.*

1.3.3 The relation between cliques and colorings

A complete subgraph of a graph G is called **clique** (that is the counterpart to a stable set). A clique containing as many nodes as possible is *maximum*, the cardinality of a maximum clique of G is the **clique number** $\omega(G)$. It gives a trivial lower bound for the chromatic number, i.e.

$$\omega(G) \leq \chi(G) \tag{1.3}$$

since in any clique all nodes should receive different colors. There are several graphs having strict inequality in (1.3). All odd holes C_{2k+1} are examples since $\omega(C_{2k+1}) = 2$ but $\chi(C_{2k+1}) = 3$ holds:



There exists a famous series of graphs $M_1, M_2, \ldots$ showing that the gap between clique and chromatic number can be arbitrarily large, too.

Construct the **Mycielski graphs** M_i for each natural number $i \geq 1$ iteratively. Let $M_1 = K_2$ and obtain M_{i+1} from M_i as follows:

- take the nodes $x_1, \ldots, x_n$ of M_i,
- add new nodes $y_1, \ldots, y_n$ and z,
- join every node y_j with all neighbors of x_j in M_i, and join z with all nodes $y_1, \ldots, y_n$.

(M_3 is also called Grötzsch graph.)

In order to establish $\omega(M_i) + i - 1 = \chi(M_i)$ for all Mycielski graphs M_i, we need:

Lemma 1.14 *Let c be a $\chi(G)$-coloring of a graph G. For every color of c, there exists a node x having this color s.t. all remaining available colors appear in $N(x)$.*

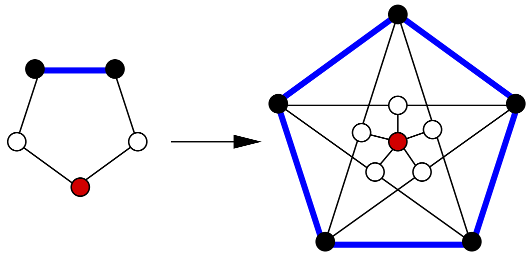


Figure 1.4: The Mycielski graphs M_2 and M_3

Proof. Assume in contrary, there is one color class S_i s.t. for every node $x \in S_i$ there is one of the remaining available colors not used in $N(x)$. Hence, we can recolor the graph by giving each $x \in S_i$ another color. This would yield a new coloring using one color (class) less than c, a contradiction since c is a $\chi(G)$-coloring of G. □

Theorem 1.15 (Mycielski 1955 [47]) *Any Mycielski graph M_i satisfies: $\omega(M_i) = 2$ and $\chi(M_i) = i + 1$.*

Proof. In order to show $\omega(M_i) = 2$ for $i \geq 1$, we prove the following claim.

Claim 1 *The neighborhood of every node in M_i is a stable set for $i \geq 1$.*

To prove this, we use induction on i. The assertion is obviously true for $M_1 = K_2$ and we suppose it is true for M_i. In order to verify it for M_{i+1}, consider the neighborhood $N_{M_{i+1}}(v)$ for all nodes v of M_{i+1}:

- $N_{M_{i+1}}(z) = \{y_1, \ldots, y_n\}$ is a stable set by construction.
- $N_{M_{i+1}}(y_j) = N_{M_i}(x_j) \cup \{z\}$ is a stable set, too, since $N_{M_i}(x_j)$ is stable by the induction hypothesis and z is not adjacent to any of the nodes $x_1, \ldots, x_n$.
- $N_{M_{i+1}}(x_j) = N_{M_i}(x_j) \cup \{y_k : x_k \in N_{M_i}(x_j)\}$ is left. $N_{M_i}(x_j)$ is stable by induction again. The nodes $y_1, \ldots, y_n$ form a stable set of M_{i+1} by construction. Moreover, we have that no node $x_l \in N_{M_{i+1}}(x_j)$ is adjacent to any $y_k \in N_{M_{i+1}}(x_j)$ (otherwise x_l and x_k had to be adjacent in M_i, a contradiction to $x_l, x_k \in N_{M_i}(x_j)$ and $N_{M_i}(x_j)$ stable by the induction hypothesis). Hence, $N_{M_{i+1}}(x_j)$ is a stable set, too. ◇

In order to prove $\chi(M_i) = i + 1$ for $i \geq 1$, we show the following claim.

Claim 2 $\chi(M_{i+1}) = \chi(M_i) + 1$.

Consider a $\chi(M_i)$-coloring c of M_i. Lemma 1.14 implies that, for every color used by c, there is a node x in M_i with this color s.t. all remaining available $\chi(M_i) - 1$ colors appear in $N_{M_i}(x)$. We need therefore, by construction, all $\chi(M_i)$ colors for the nodes $y_1, \ldots, y_n$ in every coloring of M_{i+1}. Since z is adjacent to all the nodes $y_1, \ldots, y_n$, every coloring has to use an additional color for z. Thus $\chi(M_{i+1}) > \chi(M_i)$ and the existence of a $(\chi(M_i) + 1)$-coloring of M_{i+1} imply together the assertion of the claim. $\diamond$

Thus, Claim 1 shows $\omega(M_i) = 2$ and from $\chi(M_1) = 2 (= i + 1)$ and Claim 2 follows finally $\chi(M_i) = i + 1$ for $i \geq 1$. $\square$

An immediate consequence of the Mycielski construction is therefore:

Corollary 1.16 *The lower bound $\omega(G)$ for $\chi(G)$ can be arbitrarily bad.*

Chapter 2

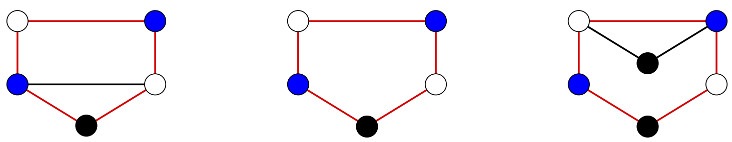
Perfect Graphs

2.1 The historical year 1960

2.1.1 The definition of perfect graphs

As we have seen by the Mycielski graphs, the gap between the lower bound $\omega(G)$ and the chromatic number $\chi(G)$ can be arbitrarily large. One is interested in graphs G where this bound is sharp. In particular, the interest focuses on graphs where this is true for *all* induced subgraphs since hereditary classes often admit a much richer structure: Berge [6] called a graph G **perfect** if and only if the clique number $\omega(G')$ coincides with the chromatic number $\chi(G')$ for all its induced subgraphs $G' \subseteq G$. All other graphs are **imperfect**.

A graph property which is closed under taking subgraphs (induced subgraphs) is called *monotone* (*hereditary*). Perfectness is a hereditary but non-monotone graph property:



perfect imperfect imperfect

Remark. The latter graph shows that an imperfect graph G may satisfy $\omega(G) = \chi(G)$ (in this case, there exists a *proper* induced subgraph $G' \subset G$ with $\omega(G') < \chi(G')$).

Perfect graphs turned out to unify several results in combinatorial optimization, in particular min-max relations and polyhedral characterizations. We will get in touch with, e.g., the integrality of polyhedra and some consequences for solving in general NP-hard problems for perfect graphs in polynomial time.

2.1.2 Some basic classes of perfect graphs

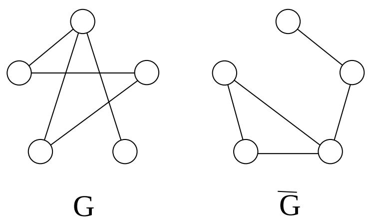
We present basic classes known to consist of perfect graphs only before 1960.

Theorem 2.1 *Every bipartite graph is perfect.*

Proof. Consider a bipartite graph $G = (A \cup B, E)$. First we show $\omega(G) = \chi(G)$. Since A and B are stable sets, we can take them as color classes and obtain $\chi(G) \leq 2$. If $\omega(G) = 1$, then E contains no edges and $\chi(G) = 1$ follows. If $\omega(G) = 2$, then $2 = \omega(G) \leq \chi(G) \leq 2$ implies $\omega(G) = \chi(G)$, too. The assertion follows since the property of being bipartite is hereditary, hence the same argumentation applies to every induced subgraph. □

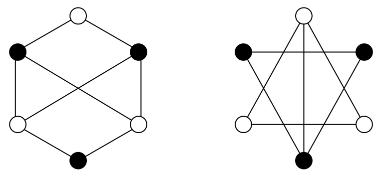
The next three classes under consideration can be constructed from bipartite graphs by applying two graph transformations: taking the complement and/or the line graph. The perfectness of those three classes follows from historical theorems due to König. Those theorems deal with min-max relations of graph parameters in bipartite graphs which can be 'translated' to the clique and chromatic number in the respective graphs.

The graph $\overline{G} = (V, V^2 - E)$ is the **complement** of $G = (V, E)$, i.e., $\overline{G}$ is the graph on the same node set as G having precisely the edges that are missing in G.



Remark. The union of G and $\overline{G}$ yields the complete graph (V, V^2). We have $G = \overline{\overline{G}}$. Furthermore, if G' is an *induced* subgraph of G then $\overline{G}'$ is an *induced* subgraph of $\overline{G}$, too (this is *not* true for other subgraphs).

The complement of a stable set is a clique (and vice versa). Hence the complement of a bipartite graph can be partitioned into two cliques:



G $\overline{\text{G}}$

The perfectness of complements of bipartite graphs follows from Königs Edge Cover Theorem. To state the theorem, we need the following notations.

A node without any neighbor is called *isolated*. The *edge cover number* $\rho(G)$ is the smallest number of edges covering all nodes of G. Since no two nodes of a stable set can be covered by the same edge, the stability number is a trivial lower bound for the edge cover number, i.e.

$$\alpha(G) \leq \rho(G) \tag{2.1}$$

This bound is sharp for, e.g., odd cycles by $\alpha(C_{2k+1}) = k$ but $\rho(C_{2k+1}) = k + 1$. König proved equality for bipartite graphs:

Theorem 2.2 (Edge Cover Theorem, König 1931 [42]) *Any bipartite graph G without isolated nodes satisfies $\alpha(G) = \rho(G)$. That is, the size of a maximum stable set in a bipartite graph equals the minimum size of an edge cover.*

In order to show the perfectness of complements of bipartite graphs, we have to translate $\alpha(G)$ and $\rho(G)$ to the complementary graph. First note that an edge cover is, in any graph without isolated nodes, a special case of a **clique cover**, that is a partition of the nodes of a graph into cliques. The minimal number of cliques needed to cover all nodes of G is the **clique cover number** $\overline{\chi}(G)$.

Since taking complements transfers cliques into stable sets and vice versa, we obtain immediately:

Proposition 2.3 *For any graph G holds $\alpha(G) = \omega(\overline{G})$ and $\overline{\chi}(G) = \chi(\overline{G})$.*

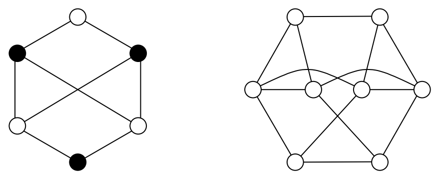
Remark. Analogously to $\omega(G) \leq \chi(G)$, there is also a min-max relation $\alpha(G) \leq \overline{\chi}(G)$ between maximum stable sets and minimum clique covers.

As a consequence, the Edge Cover Theorem implies:

Corollary 2.4 *The complements of bipartite graphs are perfect.*

Proof. Let G be a bipartite graph and G^* be the subgraph of G induced by its non-isolated nodes. Then König's Edge Cover Theorem 2.2 says $\alpha(G^*) = \rho(G^*)$. Let k stand for the number of isolated nodes of G (possibly $k = 0$). Then $\alpha(G) = \alpha(G^*) + k$ and $\overline{\chi}(G) = \rho(G^*) + k$ follows immediately and implies $\alpha(G) = \overline{\chi}(G)$. Proposition 2.3 shows, therefore, $\omega(\overline{G}) = \chi(\overline{G})$. That every induced subgraph G' of G is bipartite, too, and corresponds to an induced subgraph $\overline{G}'$ of $\overline{G}$ finishes the proof. □

Next, let's turn to line graphs constructed as follows: The **line graph** $L(G)$ of a graph G has the edges of G as nodes and two nodes are adjacent in $L(G)$ iff the corresponding edges of G are incident.



G L(G)

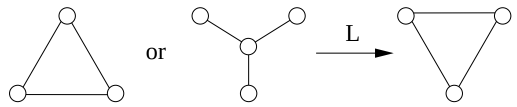
The perfectness of line graphs of bipartite graphs follows from König's Edge Coloring Theorem. To state the theorem, we need the notation of an *edge coloring*: that is a coloring of the edges of a graph G s.t. no two incident edges receive the same color. The minimal number of colors needed for that is called the *edge coloring number* (or *chromatic index*) $\gamma(G)$. Since all edges incident to a particular node of G have to be colored differently, the maximum degree of G is a trivial lower bound on $\gamma(G)$, i.e.

$$\Delta(G) \leq \gamma(G) \tag{2.2}$$

holds for any graph G. This bound is sharp for, e.g., odd cycles, by $\Delta(C_{2k+1}) = 2$ and $\gamma(C_{2k+1}) = 3$. In contrary, König proved equality for bipartite graphs:

Theorem 2.5 (Edge Coloring Theorem, König 1916 [41]) *Every bipartite graph G satisfies $\Delta(G) = \gamma(G)$. That is, the maximum degree in a bipartite graph equals the minimum number of colors needed to color all edges.*

In order to show the perfectness of line graphs of bipartite graphs, we have to translate $\Delta(G)$ and $\gamma(G)$ to the line graph $L(G)$. Obviously, an edge coloring of any graph G corresponds one-to-one to a (node) coloring of $L(G)$, hence $\gamma(G) = \chi(L(G))$ holds. Pairwise incident edges of G correspond to pairwise adjacent nodes and, therefore, cliques of $L(G)$. There are two ways to produce triangles in the line graph:

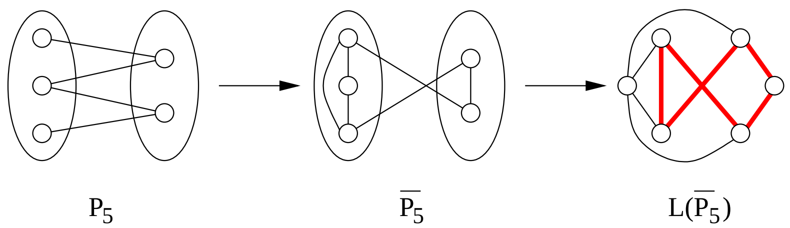


In particular, $\Delta(K_3) \neq \omega(L(K_3))$ holds. In triangle-free graphs G, however, pairwise incident edges have to form stars $K_{1,k}$ and thus $\Delta(G) = \omega(L(G))$ holds (*Prove as exercise!*).

Corollary 2.6 *Line graphs of bipartite graphs are perfect.*

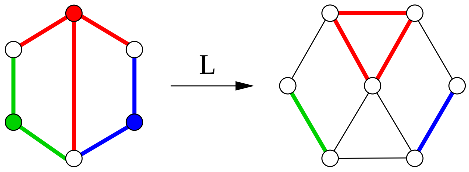
Proof. Consider a bipartite graph G. Then $\Delta(G) = \gamma(G)$ follows from König's Edge Coloring Theorem 2.5. We have $\gamma(G) = \chi(L(G))$ for any graph and $\Delta(G) = \omega(L(G))$ since G is triangle-free. Thus, $L(G)$ satisfies $\omega(L(G)) = \chi(L(G))$. All *subgraphs* G' of G are bipartite, hence the argumentation applies to all *induced subgraphs* of $L(G)$. □

Finally, turn to the third class constructed from bipartite graphs by taking complements *and* line graphs. Note that line graphs of complements of bipartite graphs are imperfect in general:



We show that complements of line graphs of bipartite graphs are perfect with the help of König's Matching Theorem 1.8. Recall that the matching number $\nu(G)$ stands for the maximum number of pairwise non-incident edges in G and the node cover number $\tau(G)$ for the minimum number of nodes meeting all edges of G.

We again have to translate these two parameters to the graph in question, namely $\overline{L(G)}$. Obviously, matchings of G correspond to stable sets of $L(G)$ and thus $\nu(G) = \alpha(L(G))$. A node cover $\mathcal{C}$ of G can be seen as a partition of G into stars (take the nodes v in $\mathcal{C}$ together with the edges covered by v) and corresponds, therefore, to a clique cover of $L(G)$:

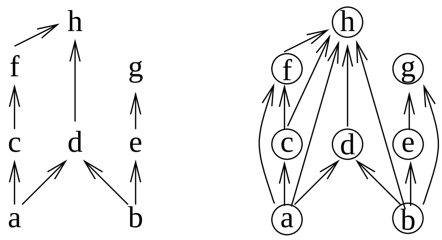


However, $\tau(K_3) \neq \overline{\chi}(L(K_3))$ holds. Again, if G is triangle-free then cliques in $L(G)$ come from stars in G only and $\tau(G) = \overline{\chi}(L(G))$ follows (*Prove as exercise!*). Proposition 2.3 finally yields:

Corollary 2.7 *Complements of line graphs of bipartite graphs are perfect.*

Proof. Consider a bipartite graph G. Then $\nu(G) = \tau(G)$ follows from König's Matching Theorem 1.8. We have $\nu(G) = \alpha(L(G))$ for any graph and $\tau(G) = \overline{\chi}(L(G))$ since G is triangle-free. Thus, $L(G)$ satisfies $\alpha(L(G)) = \overline{\chi}(L(G))$. Proposition 2.3 yields, therefore, $\omega(\overline{L(G)}) = \chi(\overline{L(G)})$. All *subgraphs* of G are bipartite, hence the argumentation applies to all *induced subgraphs* of $\overline{L(G)}$. □

There are two further graph classes which were known to be perfect in 1960. Let $(V, <)$ be a partially ordered set or, briefly a poset (i.e. V is a finite set and $<$ an antisymmetric and transitive relation). Two elements $x, y \in V$ are *comparable* if either $x < y$ or $y < x$ holds and *incomparable* otherwise. $G = (V, E)$ with $E = \{xy : x, y \text{ comparable w.r.t } <\}$ is called the **comparability graph** corresponding to the poset $(V, <)$.



(V,<) G=(V,E)

Exercise. Prove that every bipartite graph is a comparability graph.

Pairwise comparable (resp. incomparable) elements of a poset are called a *chain* (resp. *antichain*). We easily obtain:

Theorem 2.8 *In any poset $(V, <)$, as many antichains cover V as a maximum chain has elements.*

Proof. Denote by $l(x)$ the number of elements of a largest chain with end-element x and construct sets $S_i := \{x \in V : l(x) = i\}$ for $1 \leq i \leq l(V) = \max\{l(x) : x \in V\}$. Every set S_i is an antichain of $(V, <)$ since $x < y$ for two elements $x, y \in S_i$ would imply $l(y) \geq l(x) + 1 = i + 1$, a contradiction to the construction of S_i. Then $S_1, \ldots, S_{l(V)}$ are antichains covering V by construction and, moreover, there is a chain of size $l(V)$. □

Chains resp. antichains of a poset $(V, <)$ correspond to cliques resp. stable sets of the comparability graph $G = (V, E)$. Hence, Theorem 2.8 implies:

Corollary 2.9 *Every comparability graph is perfect.*

Proof. The number of elements $l(V)$ in a longest chain of $(V, <)$ equals obviously the clique number $\omega(G)$ of G. Hence, the antichains $S_1, \ldots, S_{l(V)}$ covering V yield an $\omega(G)$-coloring of G due to Theorem 2.8. Applying the same argumentation to every $V' \subseteq V$ yields $\omega(G') = \chi(G')$ for all induced subgraphs $G' \subseteq G$. □

The perfectness of complements of comparability graphs is an immediate consequence of the following theorem (*Proof as exercise*):

Theorem 2.10 (Dilworth 1950 [27]) *In a poset $(V, <)$, the size of a maximum antichain equals the minimal number of chains needed to cover V.*

2.1.3 Two historical conjectures

Claude Berge observed in 1960 firstly that, for all the known classes of perfect graphs, both the class itself and the complementary class are perfect. This let him conjecture:

Conjecture 2.11 (Perfect Graph Conjecture (PGC), Berge 1960 [6]) *A graph G is perfect if and only if its complement $\overline{G}$ is perfect.*

Remark. The PGC tells that perfect graphs G satisfy *two* min-max relations of graph parameters, namely $\omega(G') \leq \chi(G')$ and $\alpha(G') \leq \overline{\chi}(G')$, with equality for all induced subgraphs G' of G.

Secondly, Berge thought about (minimal) forbidden subgraphs in perfect graphs. He knew that all odd holes $C_{2k+1}, k \geq 2$ are imperfect by $\omega(C_{2k+1}) = 2$ but $\chi(C_{2k+1}) = 3$. With look at the PGC, Berge observed furthermore that the complements of odd holes, termed **odd antiholes** $\overline{C}_{2k+1}, k \geq 2$, are imperfect, too (*Proof as exercise*).

Hence, neither odd holes nor odd antiholes can appear as induced subgraphs of any perfect graph. Berge's famous conjecture was that it *suffices* to forbid these two kinds of induced subgraphs in perfect graphs:

Conjecture 2.12 (Strong Perfect Graph Conjecture (SPGC), Berge 1960 [6]) *A graph is perfect if and only if it has no odd hole or odd antihole as induced subgraph.*

2.2 The Perfect Graph Theorem and minimally imperfect graphs

In order to prove the PGC, Lovàsz reformulated the conjecture as follows:

Conjecture 2.13 (Lovàsz 1972 [44]) *A graph G is perfect if and only if $\alpha(G')\omega(G') \geq |G'|$ holds for all induced subgraphs $G' \subseteq G$.*

Obviously, the above condition is invariant under taking complements, hence Conjecture 2.13 implies the PGC.
Prove as exercise: Conjecture 2.13 is equivalent to the PGC.

The idea for verifying Conjecture 2.13 was to show that $\alpha(G)\omega(G) < |G|$ holds for all **minimally imperfect graphs** G: imperfect graphs having perfect graphs as *proper* induced subgraphs only.

Remark. Odd holes resp. odd antiholes are *minimally* imperfect since they are imperfect but all their proper induced subgraphs are bipartite resp. complements of bipartite graphs. The SPGC says in this terms that the odd holes and odd antiholes are the *only* minimally imperfect graphs.

In order to prove the above property for minimally imperfect graphs, we need:

Lemma 2.14 *A minimally imperfect graph G satisfies $\omega(G - S) = \omega(G)$ for every stable set S of G.*

Proof. Consider an arbitrary stable set S of a minimally imperfect graph G. Suppose S meets all maximum cliques Q of G. Then $\omega(G-S) = \omega(G) - 1$ holds due to $|S \cap Q| = 1$ for all Q. The proper induced subgraph $G - S \subset G$ is perfect and admits, therefore, an $\omega(G-S)$-coloring. Adding one color class, namely S, yields an $\omega(G)$-coloring of G, a contradiction to $\omega(G) < \chi(G)$ due to G minimally imperfect. $\square$

Theorem 2.15 (Lovàsz 1972 [44]) *Minimal imperfect graphs G satisfy $|G| = \alpha(G)\omega(G) + 1$.*

Proof (due to Gasparian 1996 [34]). Let G be minimally imperfect and denote by α, ω resp. n the values of $\alpha(G)$, $\omega(G)$ resp. $|G|$. We prove the assertion with the help of the following three claims.

Claim 1 *$G - v$ admits, for every node v, an ω-coloring.*

Lemma 2.14 implies $\omega(G-v) = \omega(G)$. Moreover, $G - v$ is perfect and can, therefore, be colored using $\omega(G-v) = \omega$ colors. $\diamond$

Consider a maximum stable set S_0 in G. Let $S_1, \ldots, S_{\alpha\omega}$ be the color classes of ω-colorings of $G - v$, for all $v \in S$. (This construction is possible by Claim 1: let $S_1, \ldots, S_\omega$ be the color classes of $G - v_1$, $S_{\omega+1}, \ldots, S_{2\omega}$ the color classes of $G - v_2$, etc.) Moreover, let Q_i be a maximum clique (of size ω) of G in $G - S_i$ for $0 \le i \le \alpha\omega$ (these cliques exist due to Lemma 2.14).

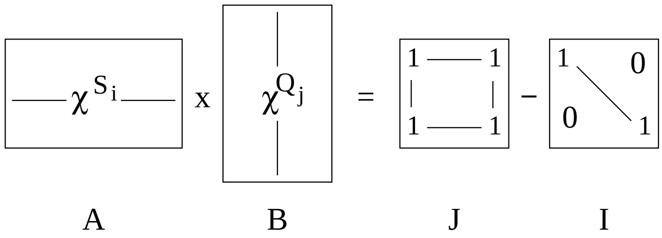
Claim 2 *$|Q_i \cap S_j| = 1$ for $i \neq j$.*

Consider an arbitrary maximum clique Q of G.

If $Q \cap S_0 = \emptyset$ then $Q \subseteq G - v$ for all $v \in S_0$. Thus Q intersects every of the ω color classes of $G - v$ for all $v \in S_0$ by Claim 1, hence $|Q \cap S_j| = 1$ for $1 \le j \le \alpha\omega$ and $Q = Q_0$.

Otherwise, we have $Q \cap S_0 = \{v_0\}$. Then Q does not intersect precisely one of the ω color classes of $G - v_0$, say S_k, but Q intersects all ω color classes of $G - v$ for all $v \in S_0 - v_0$. Hence $|Q \cap S_j| = 1$ for $k \neq j$ and $Q = Q_k$. $\diamond$

Let A be the $(\alpha\omega+1) \times n$-matrix having the incidence vectors of $S_0, \ldots, S_{\alpha\omega}$ as rows and B be the $n \times (\alpha\omega + 1)$-matrix having the incidence vectors of $Q_0, \ldots, Q_{\alpha\omega}$ as columns. Furthermore, let J stand for the matrix with 1-entries only and I for the identity matrix, both of dimension $(\alpha\omega + 1) \times (\alpha\omega + 1)$.



Claim 3 $AB = J - I$.

The matrix AB has 0-entries on the main diagonal (since $Q_i \cap S_i = \emptyset$ by construction) and 1-entries elsewhere (since $|Q_i \cap S_j| = 1$ for $i \neq j$ holds by Claim 2). $\diamond$

On the one hand, $G - v$ can be partitioned into ω stable sets, each of size $\leq \alpha$, thus we have $n \leq \alpha\omega + 1$. On the other hand, Claim 3 shows that AB is regular since $J - I$ is. Hence A must have full row-rank. That A is an $(\alpha\omega + 1) \times n$-matrix implies $\alpha\omega + 1 \leq n$. This finally shows $\alpha\omega + 1 = n$. $\square$

Corollary 2.16 (Lovàsz 1972 [44]) *A graph G is perfect if and only if $\alpha(G')\omega(G') \geq |G'|$ holds for all induced subgraphs $G' \subseteq G$.*

Proof. If G is a perfect graph, then every of its induced subgraphs G' admits an $\omega(G')$-coloring with color classes of size $\leq \alpha(G')$. Thus $\omega(G') = \chi(G')$ and $\chi(G') \geq \frac{|G'|}{\alpha(G')}$ implies $\alpha(G')\omega(G') \geq |G'|$.

Conversely, if $\alpha(G')\omega(G') \geq |G'|$ is true for every induced subgraph G' of G, then none of the induced subgraphs of G is minimally imperfect due to Theorem 2.15. $\square$

Corollary 2.17 (Perfect Graph Theorem (PGT), Lovàsz 1972 [44]) *A graph is perfect if and only if its complement is perfect.*

Proof. Corollary 2.16 is invariant under taking complements. $\square$

Remark. The PGT generalizes the three historical theorems on bipartite graphs due to König and Dilworth theorem on posets.

2.3 The Strong Perfect Graph Theorem and Berge Graphs

Chvátal & Sbihi supposed to call, in honor of Claude Berge, a graph **Berge** iff it has no odd hole or odd antihole as induced subgraph.

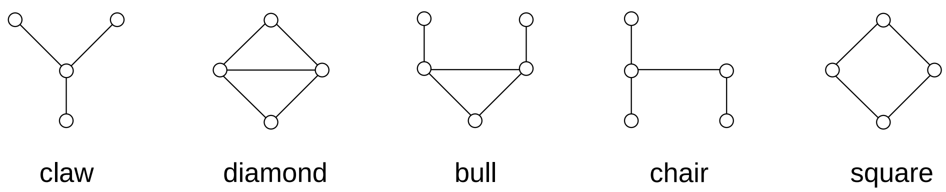
Using this term, the SPGC says that every Berge graph is perfect. A counterexample to the SPGC would, therefore, be a minimal imperfect Berge graph (sometimes called 'monster').

Considerable effort has been spent over the years to verify or falsify the SPGC. Thereby, many structural properties of minimally imperfect graphs have been discovered and the conjecture has been proved for many classes of Berge graphs.

2.3.1 Classes of perfect Berge graphs

The SPGC has been verified for several graph classes, e.g., for

- claw-free graphs (Parthasarathy & Ravindra 1976 [51])
- K_4-free graphs (Tucker 1977 [58])
- diamond-free graphs (Tucker 1987 [59])
- bull-free graphs (Chvátal & Sbihi 1987 [16])
- chair-free graphs (Sassano 1997 [55])
- square-free graphs (Conforti, Cornuéjols & Vušković 2001 [20])

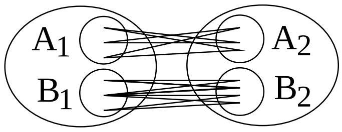


The latter result is due to the following decomposition theorem:

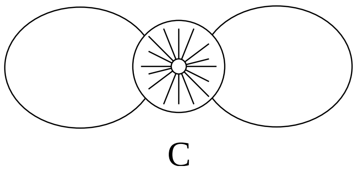
Theorem 2.18 (Conforti, Cornuéjols & Vušković 2001 [20]) *Every square-free Berge graph G satisfies one of the following conditions:*

(i) *G is bipartite.*
(ii) *G is the line graph of a bipartite graph.*
(iii) *G has a 2-join or a star-cutset.*

A **2-join** of a graph $G = (V, E)$ is a partition $V = V_1 \cup V_2$ with non-empty subsets $A_i, B_i \subseteq V_i$ for $i = 1, 2$ s.t. both pairs (A_1, A_2) and (B_1, B_2) are completely joined but there are no other edges between V_1 and V_2.



A **star-cutset** is a set C of nodes of a graph G s.t. $G - C$ is disconnected ('cutset') and there is a node in C which is adjacent to all the remaining nodes of C ('star').



Due to the following two theorems, neither 2-joins nor star-cutsets appear in minimally imperfect Berge graphs.

Theorem 2.19 (Cornuéjols & Cunningham 1985 [23]) *Every minimally imperfect graph with a 2-join is an odd hole.*

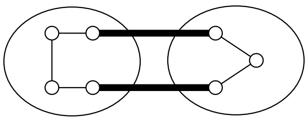


Figure 2.1: A 2-join in an odd hole

Theorem 2.20 (Chvátal 1985 [15]) *No minimally imperfect graph has a star-cutset.*

Putting all pieces together, the Decomposition Theorem 2.18 implies that every square-free Berge graph is either perfect or has a structural fault which does not occur in minimally imperfect Berge graphs. This finally implies:

Corollary 2.21 (Conforti, Cornuéjols & Vušković 2001 [20]) *Square-free Berge graphs are perfect.*

This result motivated Conforti, Cornuéjols & Vušković to ask whether the class of *all* Berge graphs can be decomposed in a similar way in order to prove the SPGC:

Conjecture 2.22 (Decomposition Conjecture, Conforti, Cornuéjols & Vušković 2001 [20]) *Every Berge graph G satisfies one of the following conditions:*

(i) *G or $\overline{G}$ is bipartite or the line graph of a bipartite graph.*

(ii) *G or $\overline{G}$ admits structural faults as, e.g., 2-joins, star-cutsets, or skew partitions.*

A **skew partition** of a graph $G = (V, E)$ is a partition $V = A \cup B \cup C \cup D$ into non-empty subsets s.t. the pair (A, B) is completely joined and the pair (C, D) is completely unjoined.

2.3.2 Decomposing Berge graphs and consequences

We outline the structure of the proof for the SPGC (a preliminary version can be found at `http://www.math.gatech.edu/~thomas/spgc.html`).

Based on the Decomposition Conjecture 2.22, Chudnovsky, Robertson, Seymour & Thomas [12] were able to prove in 2002 a slightly different decomposition for Berge graphs, involving one more kind of basic perfect graphs and different structural faults:

Theorem 2.23 (Chudnovsky, Robertson, Seymour & Thomas [12]) *Every Berge graph G satisfies one of the following conditions:*

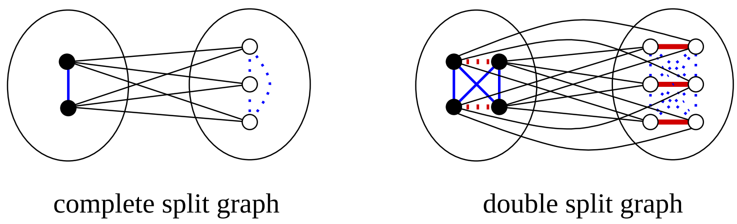
(i) *G or $\overline{G}$ is bipartite.*
(ii) *G or $\overline{G}$ is the line graph of a bipartite graph.*
(iii) *G is a double-split graph.*

(iv) *G or $\overline{G}$ has a 2-join.*
(v) *G has an M-join.*
(vi) *G has a balanced skew partition.*

Double-split graphs, M-joins, and balanced skew partitions are invariant under taking complements and are defined as follows.

To construct a **double-split graph**,

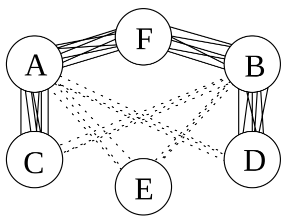
- take a *complete split graph* $G = (Q \cup S, E)$ where Q is a clique, S a stable set, and E contains all possible edges between Q and S;
- replace every node x_i in Q by two non-adjacent nodes x_i', x_i'', replace every node y_j in S by two adjacent nodes y_j', y_y'';
- split the edge between x_i and y_j into two edges s.t. the four nodes x_i', x_i'', y_j', y_y'' induce a P_4.



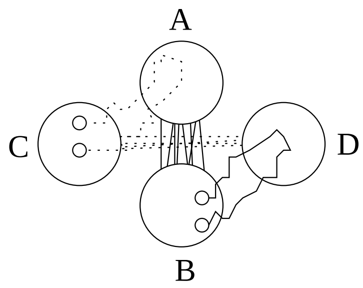
Exercise. Show that every double-split graph is perfect. (Note: split graphs are perfect.)

An **M-join** of a graph $G = (V, E)$ is a partition $V = A \cup B \cup C \cup D \cup E \cup F$ into non-empty subsets s.t.

- the pairs (C, A), (A, F), (F, B), (B, D) are completely joined,
- the pairs (D, A), (A, E), (E, B), (B, C) are completely unjoined,
- every node in A has in B both a neighbor and a non-neighbor, and vice versa.



Finally, a skew partition $V = A \cup B \cup C \cup D$ of a graph $G = (V, E)$ is **balanced** if there is no odd induced path in G (resp. $\overline{G}$) having its endnodes in $A \cup B$ (resp. $C \cup D$) and its inner nodes in $C \cup D$ (resp. $A \cup B$).



Theorem 2.24 (Chudnovsky, Robertson, Seymour & Thomas [12]) *No minimally imperfect graph has an M-join. No minimum counterexample to the SPGC has a balanced skew partition.*

Remark. At the moment, it is open whether (general) skew partitions can occur in minimally imperfect Berge graphs.

Putting all the pieces together, the above Decomposition Theorem 2.23 implies that every Berge graph is either perfect or has a structural fault which does not occur in a *minimum* counterexample to the SPGC. Hence, there is *no* counterexample to the SPGC:

Theorem 2.25 (Strong Perfect Graph Theorem (SPGT), Chudnovsky, Robertson, Seymour & Thomas 2002 [12]) *Every Berge graph is perfect.*

Remark. We conclude with some notes on decomposing Berge graphs. Chudnovski [10] showed in her PhD thesis that M-joins can be dropped from the Decomposition Theorem 2.23. Double-split graphs admit (general) skew partitions, hence proving (without using the SPGT) that no minimally imperfect Berge graph has a (general) skew partition would prove the original Decomposition Conjecture 2.22.

2.4 Recognizing perfect graphs

Deciding whether an arbitrary graph is perfect was known to be in co-NP: examining one (minimally) imperfect induced subgraph suffices to decide that the (whole) graph is not perfect. However, the following problem was open:

Problem 2.26 (Recognition problem) *Is there a polynomial time algorithm to test whether an arbitrary graph is perfect?*

Due to the SPGT, the Recognition Problem 2.26 is equivalent to test in polynomial time whether a given graph G is Berge. This could be done by deciding whether G or $\overline{G}$ admits an odd hole as induced subgraph.

However, the problem to test whether a graph contains an odd hole passing through a specific node is NP-complete by Bienstock 1991 [7] and the existence of a polynomial time algorithm *to decide whether an arbitrary graph contains an odd hole is still open!*

Remark. Conforti, Cornuéjols, Kapoor & Vušković [18] designed in 2002 a polynomial time algorithm to test for *even* holes.

Surprisingly, Chudnovsky & Seymour and Cornuéjols, Liu & Vušković were able to design polynomial time algorithms for recognizing Berge graphs and, therefore, perfect graphs, see [13, 11, 24]. The two algorithms have to be run for G and $\overline{G}$ and output 'the graph is not *Berge* or else it contains no odd hole'.

The common idea for these algorithms is to use the concept of 'clean graphs' (due to Conforti & Rao [22]) where the *shortest* odd hole has a certain property.

Remark. The strategy of *cleaning* was first introduced by Conforti & Rao [22] in 1992 to recognize linear balanced matrices. It was also used in the recognition algorithm for balanced matrices (by Conforti, Cornuéjols & Rao [19] in 1999 resp. Conforti, Cornuéjols, Kapoor & Vušković [17] in 2001) and even hole-free graphs (by Conforti, Cornuéjols, Kapoor & Vušković [18] in 2002).

In a joint work, Chudnovsky, Cornuéjols, Liu, Seymour & Vušković [11] designed in 2002 a polynomial time algorithm 'cleaning for Bergeness' as first step of a recognition algorithm.

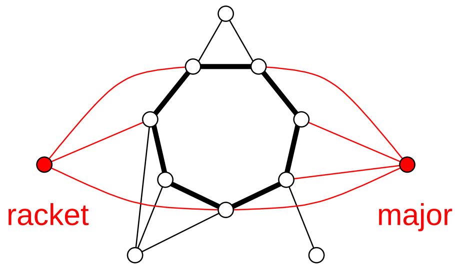
There exist different second steps for the algorithm, i.e., two ways to decide in polynomial time whether a 'clean graph' contains an odd hole:

- Cornuéjols, Liu & Vušković [24] designed an algorithm based on the decomposition for odd hole-free graphs due to Conforti, Cornuéjols & Vušković [21] in 2001 (note: those decompositions preserve odd hole-freeness only in clean graphs!)
- Chudnovsky & Seymour [13] developed a direct tool, called 'clean shortest odd hole detector' not using decomposition at all.

2.4.1 Cleaning for Bergeness

The interest focuses on a *shortest* odd hole in a graph since this provides additional structural properties.

Let C be a shortest odd hole in a graph $G = (V, E)$. Call a node **major** for C if its neighbors on C do not lie in a 2-edge path of C. A major node has at least 3 neighbors on C and is a *racket* if it has precisely 3 neighbors on C.



In order to ‘clean‘ the graph G, we have to remove all major nodes for C. A subset $X \subseteq V$ is a **cleaner** for C if X contains every major node for C and $X \cap V(C)$ belongs to a 2-edge-path.

Although one does not know a shortest odd hole C and there are, at first sight, exponentially many possibilities for C, it is possible to enumerate polynomially many sets $X \subseteq V$ s.t. one of the sets X is guaranteed to be a cleaner for C if C exists at all:

Theorem 2.27 (Chudnovsky, Cornuéjols, Liu, Seymour & Vušković 2002 [11]) *There is an $O(|V|^6)$-algorithm which outputs, for any graph $G = (V, E)$, either $O(|V|^5)$ subsets of V s.t. one of the sets is a cleaner for C if C is a shortest odd hole with no racket, or determines that G is not Berge.*

Remark. “There are places in the algorithm where we stumble over an odd antihole and stop, declaring the graph non-Berge, and we are currently unable to eliminate that; so the question of testing just for odd holes remains open.” Chudnovsky et al. [11]

2.4.2 The approach due to Cornuéjols, Liu & Vušković

To test in *clean* graphs for odd holes, Cornuéjols, Liu & Vušković decompose in [24] a graph G into a polynomial number of simpler graphs $G_1, \dots, G_k$ s.t.

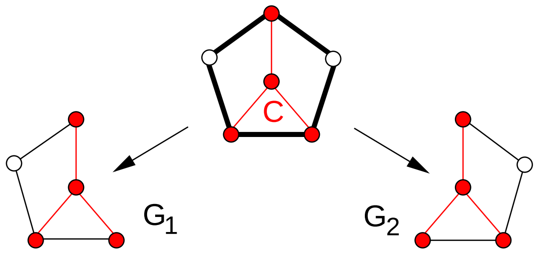
- G is odd hole-free iff G_i is odd hole-free and
- it is easy to check for odd holes in G_i

for all $1 \leq i \leq k$. In order to satisfy the first condition, the graph is supposed to be clean. The decomposition into blocks bases on:

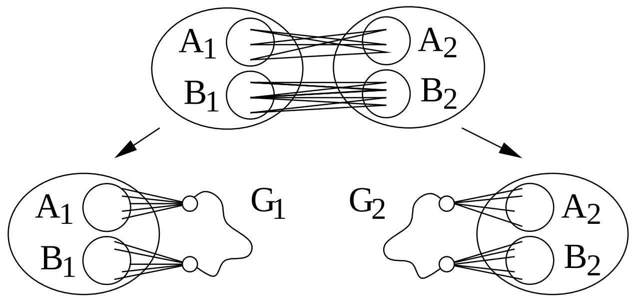
Theorem 2.28 (Decomposition Theorem, Conforti, Cornuéjols & Vušković [21] 2001) *Every odd hole-free graph G satisfies one of the following:*

(i) *G is bipartite.*
(ii) *G or $\overline{G}$ is the line graph of a bipartite graph.*
(iii) *G has a double-star cutset or a proper 2-join.*

Here a cutset C of a graph G is called a **double-star** if C contains two adjacent nodes x and y with $C \subseteq N(x) \cup N(y)$. Let $G_1, \dots, G_k$ be the components of $G - C$, then the subgraphs of G induced by $G_i \cup C$ are the *blocks* of the decomposition. The double-star decomposition preserves odd hole-freeness in *clean* graphs only:



A 2-join $V_1|V_2$ is *proper* if none of the two parts V_1 and V_2 induces a chordless path. The *blocks* of the decomposition are obtained as follows and preserve odd hole-freeness in any graph. Let P be a shortest path connecting A_2 and B_2 in $G[V_2]$. We define the block G_1 to be the subgraph $G[V_1 \cup P_5]$ resp. $G[V_1 \cup P_6]$ where P is replaced by a P_5 resp. P_6 if P is even resp. odd; otherwise let G_1 be $G[V_1 \cup \{a, b\}]$ where a is completely joined to any node in A_1 and b to any node in B_1. The block G_2 is defined similarly. (Note that the 2-join decomposition of an odd hole would yield two odd holes as blocks. Hence considering *proper* 2-joins only provents from duplicating odd holes during the decomposition.)



Algorithm 2.29 (Decomposing clean graphs, Cornuéjols, Liu & Vušković 2002 [24]) *Input a (clean) graph.*

- (i) *Perform double-star decompositions until no block has a double-star cutset anymore.*
- (ii) *Perform proper 2-join decompositions without creating new double-star cutsets, until no proper 2-joins exist either.*
- (iii) *Check whether the obtained blocks are bipartite, line graphs of bipartite graphs, or complements of the latter.*

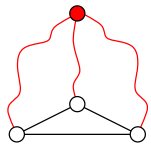
Remark. In a clean graph, the blocks of the decomposition obtained in the first two steps of Algorithm 2.29 contain an odd hole if and only if the original graph G contains an odd hole. If G is odd hole-free, then Theorem 2.28 guarantees that all blocks belong to one of the three basic classes of perfect graphs for which Algorithm 2.29 tests in the third step.

Theorem 2.30 (Cornuéjols, Liu & Vušković [24]) *The Algorithm 2.29 decides in $O(|V|^{10})$ whether a clean graph $G = (V, E)$ contains an odd hole.*

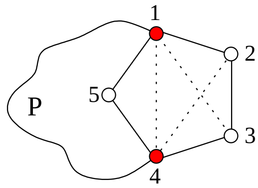
2.4.3 The approach by Chudnovsky & Seymour

In order to detect clean shortest odd holes, Chudnovsky & Seymour take a clean graph as input and search firstly for two configurations, called pyramids and jewels. If they exist, then both configurations contain an easily detectable odd hole. The algorithm finally looks for odd holes in clean pyramid- and jewel-free graphs (using no decomposition at all).

Pyramids are subdivisions of a K_4 where the three edges incident to *one* of the nodes are allowed to be subdivided and at most one of these edges remains undivided. At least two of the arising paths have the same parity, hence every pyramid contains an odd hole. (Pyramid-free graphs are also called 3PC(.,.)-free graphs by Aossey & Vušković, unpublished yet.)



A **jewel** is a 5-cycle $v_1, \ldots, v_5$ with non-edges v_1v_3, v_2v_4, v_1v_4 together with a chordless path P linking the nodes v_1 and v_4 s.t. v_2, v_3, v_5 have no neighbors among the inner nodes of P. Every jewel contains an odd hole: either $P \cup \{v_5\}$ if P has odd length or $P \cup \{v_2, v_3\}$ if P has even length. (Note: P has length ≥ 2 due to v_1, v_4 non-adjacent.)



Algorithm 2.31 (Clean shortest odd hole detector, Chudnovsky & Seymour 2002 [13]) *Input a (clean) graph G.*

(i) *Find either a pyramid (and hence an odd hole) in G and stop, or determine that G contains no pyramid.*

(ii) *Decide whether there is a jewel (and hence an odd hole) in G and stop, or claim that G is jewel-free.*

(iii) *Find an odd hole in G or output that there is no clean shortest odd hole in G.*

Theorem 2.32 (Chudnovsky & Seymour 2002 [13]) *Algorithm 2.31 decides in $O(|V|^9)$ whether a clean graph $G = (V, E)$ contains an odd hole.*

Chapter 3

The Stable Set Polytope and its LP-Relaxations

3.1 IP's and LP-relaxations

Let $A \in \mathbb{R}^{m,n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The program

$$\max c^T x, Ax \leq b, x \in \mathbb{Z}^n \tag{3.1}$$

is an **integer program (IP)**. It is NP-complete to decide whether (3.1) has a solution above a certain threshold. In contrary, a **linear program (LP)**

$$\max c^T x, Ax \leq b, x \in \mathbb{R}^n \tag{3.2}$$

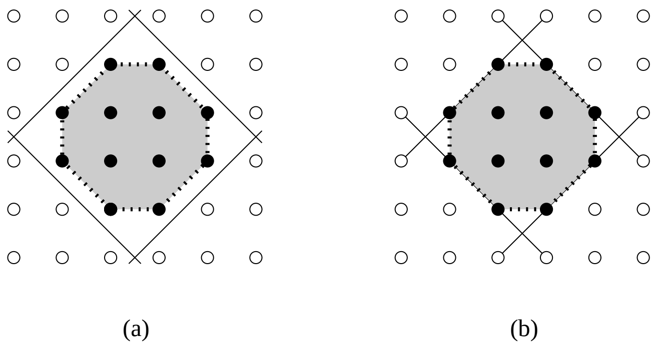


Figure 3.1: Two LP-relaxations of an IP

can in principal be solved in polynomial time. The LP (3.2) has been termed an **LP-relaxation** of the IP (3.1) since the solution space $P(A,b)$ of (3.2) contains the convex hull $P_I(A,b)$ of all (integer) solutions of (3.1).

How can we benefit from the polynomial time solvability of LP's for solving IP's? One way to do that is to look for LP-relaxations of the IP (3.1) satisfying

$$P(A,b) = P_I(A,b) \tag{3.3}$$

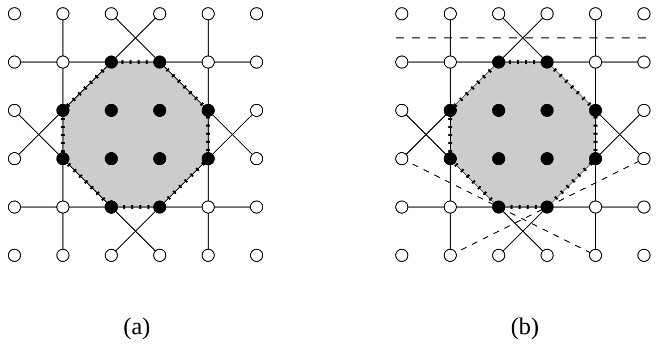
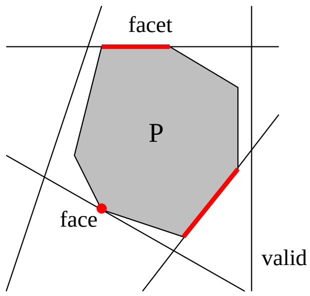


Figure 3.2: Two representations of $P_I(A,b)$ as solution space of an LP

Every row $A_i x \leq b_i$ of the system $Ax \leq b$ is a **valid inequality** of $P_I(A,b)$ due to $P_I(A,b) \subseteq \{x \in \mathbb{R}^n : A_i x \leq b_i\}$. If $P(A,b) = P_I(A,b)$ then the system $Ax \leq b$ *contains* all facet inducing inequalities of $P_I(A,b)$ (and possibly further redundant inequalities as the dashed lines in Figure 3.2(b)). Here a valid inequality $A_i x \leq b_i$ defines a **facet** of a polytope P if $\dim(\{x \in P : A_i x \leq b_i\}) = \dim(P) - 1$.



Remark. Every IP can be seen as a compact formulation of an LP: we use the integrality constraint in order to describe the solution space $P_I(A,b)$ with fewer inequalities (see Figure 3.1 and Figure 3.2).

In order to benefit from the polynomial time solubility of LP's for IP's we have, however, to describe $P_I(A,b)$ by valid inequalities s.t. the integrality constraint is superfluous. The aim is, therefore, to find the smallest system $A'x \le b'$ with $P(A',b') = P_I(A,b)$, i.e., the set of all facet inducing inequalities of $P_I(A,b)$ (as in Figure 3.2(a)).

3.2 The stable set polytope STAB(G)

The stability number $\alpha(G)$ of a graph G is defined as

$$\alpha(G) = \max\{|S| : S \subseteq G \text{ stable}\}. \tag{3.4}$$

Let $\chi^S \in \mathbb{R}^{|G|}$ with

$$\chi_i^S := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

denote the **incidence vector** of $S \subseteq G$ and call the convex hull of the incidence vectors χ^S of all stable sets S of the graph G

$$\text{STAB}(G) := \text{conv}\{\chi^S : S \subseteq G \text{ stable}\} \tag{3.5}$$

the **stable set polytope**. Obviously, STAB(G) is the solution space of (3.4): finding a maximum stable set S^* of G is equivalent to looking for a 0/1-vector χ^{S^*} in STAB(G) maximizing $\mathbb{1}^T\chi^{S^*}$ where $\mathbb{1} = (1,\ldots,1)$. Hence, determining $\alpha(G)$ corresponds to solving the IP

$$\max \mathbb{1}^T x, x \in \text{STAB}(G) \tag{3.6}$$

and we, therefore, look for LP-relaxations

$$\max \mathbb{1}^T x, Ax \le b, x \in \mathbb{R}^{|G|} \tag{3.7}$$

of the IP (3.6) satisfying

$$\text{STAB}(G) = P(A,b) \tag{3.8}$$

This is equivalent to the problem:

Problem 3.1 *Investigate whether, for certain graphs G, an LP-relaxation of (3.6) contains all facet-defining inequalities of* STAB(G).

In order to decide whether a valid inequality is a facet of STAB(G) we need to know the dimension of the polytope. It is easy to obtain

$$\dim(\text{STAB}(G)) = |G| \tag{3.9}$$

since STAB(G) contains the $|G|+1$ affinely independent vectors $\chi^{\emptyset}$ and $\chi^{\{i\}}$ for all nodes $i \in G$. Hence every facet $A_i x \leq b_i$ of STAB(G) has to contain $|G|$ affinely independent incidence vectors of stable sets S satisfying $A_i x \leq b_i$ at equality, i.e., $A_i \chi^S = b_i$. We call such a stable set S a **root** of $A_i x \leq b_i$.

Among these facets are, of course, the "trivial" facets demanding nonnegativity:

$$x_i \geq 0 \;\forall\, i \in V. \tag{3.10}$$

Obviously, there are $|G|$ roots of the nonnegativity constraint (3.10) with affinely independent incidence vectors, namely $\chi^{\emptyset}$ and $\chi^{\{j\}}$ for all nodes $j \in G - i$.

The task is to identify the remaining "nontrivial" facets

$$\sum_{i \in G} a_i\, x_i \leq b$$

where $a_1, ..., a_n$ are nonnegative integers and b is a positive integer. We often write $x(G,a) \leq b$ for such an inequality, where we interpret the vector $a = (a_1, \ldots, a_n)$ to be a node weighting of G associating the weight a_i to the node $i \in V$ for $1 \leq i \leq n$ and denote the weighted graph by (G,a). (If $a = \mathbb{1}$, we still write G instead of $(G, \mathbb{1})$.) Since $\chi^{\emptyset}$ cannot satisfy any inequality with $b > 0$ with equality, there have to exist n roots of any nontrivial facet with *linearly* independent incidence vectors.

3.3 The edge polytope ESTAB(G)

Let's start a first 'naive' try to find an LP-relaxation of STAB(G).

Consider an arbitrary graph $G = (V, E)$. The definition of STAB(G) implies immediately that the **edge constraints**

$$x_i + x_j \leq 1 \;\forall\, ij \in E \tag{3.11}$$

are valid inequalities for STAB(G) (since the incidence vector χ^S of any stable set S of G satisfies these inequalities). We infer

$$\text{STAB}(G) = \text{conv}\{x \in \{0,1\}^{|G|} : x \text{ satisfies (3.11)}\}$$

and thus

$$\max \mathbb{1}^T x, x_i + x_j \le 1 \; \forall ij \in E, \; 0 \le x_i \le 1 \; \forall i \in V, \; x \in \mathbb{R}^{|G|} \tag{3.12}$$

is an LP-relaxation of (3.6) (we only added $x_i \le 1$ for possibly isolated nodes of G). The solution space of (3.12) is called the **edge polytope**

$$\text{ESTAB}(G) = \{x \in \mathbb{R}^{|G|} : 0 \le x \le 1, \; x \text{ satisfies (3.11)}\}.$$

We have clearly $\text{STAB}(G) \subseteq \text{ESTAB}(G)$ and are interested in those graphs G for which equality holds, i.e., for which $\text{ESTAB}(G)$ contains all facets of $\text{STAB}(G)$.

For that, we need the following notation. A matrix A is called **totally unimodular** if each square submatrix of A has a determinant equal to 0, +1, or −1. In particular, each entry of a totally unimodular matrix equals 0, +1, or −1.

The total unimodularity of matrices has turned out to be an important tool in integer programming due to the following reason:

Theorem 3.2 (Hoffman & Kruskal 1956 [38]) *Let $A \in \mathbb{R}^{m,n}$ be totally unimodular, $b \in \mathbb{Z}^m$ and $P(A,b)$ be bounded. Then $P(A,b) = P_I(A,b)$.*

Viewing at the above problem, it is of interest to establish for which graphs the edge/node incidence matrix (i.e. the matrix containing the incidence vectors of all edges as rows) is totally unimodular.

Theorem 3.3 *The edge/node incidence matrix of bipartite graphs is totally unimodular.*

Proof. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph and $A \in \mathbb{R}^{|V|,|E|}$ its edge/node incidence matrix where the columns corresponding to the nodes in V_1 are listed before the columns corresponding to the nodes in V_2.

We show the assertion by induction on the size of a square submatrix A' of A. The case of 1×1-submatrices is trivial (since A has 0/1-entries only). Suppose the assertion is true for all $k \times k$-submatrices of A. In order to verify it for all $(k+1) \times (k+1)$-submatrices A', we distinguish three cases (note: each row of A contains precisely two 1-entries).

Case 1 *A' has a 0-row.*

Then we clearly have $\det(A') = 0$. $\diamond$

Case 2 *A' admits a row A'_i containing precisely one 1-entry $a_{ij} = 1$.*
Let A'' be the $k \times k$-submatrix obtained by removing the i-th row and the j-th column of A'. Developing the determinant w.r.t. the row A'_i yields

$$\det(A') = a_{ij}\ (-1)^{i+j}\ \det(A'')$$

and $\det(A'') \in \{0, +1, -1\}$ by the induction hypothesis implies $\det(A') \in \{0, +1, -1\}$, too. $\diamond$

Case 3 *Every row of A' has two 1-entries.*
Due to the bipartition of G, we have a partition of the columns of A' as $A' = (A'_1|A'_2)$ where the columns in A'_1 resp. A'_2 correspond to nodes in V_1 resp. V_2. Furthermore, each edge $ij \in E$ has one endnode $i \in V_1$ and the other endnode $j \in V_2$. Adding up all columns of A'_1 resp. A'_2 yields a vector with all 1-entries in both cases. Thus A' is singular and $\det(A') = 0$. $\diamond$

Consequently, $\det(A') \in \{0, +1, -1\}$ follows for each square submatrix A' of A and, hence, A is totally unimodular. $\square$

If A is totally unimodular then $\begin{pmatrix} A \\ I \end{pmatrix}$ is totally unimodular where I stands for the identity matrix. The constraint matrix defining the edge polytope consists in the edge/node incidence matrix of the graph and two identity matrices; hence it is totally unimodular in the case of bipartite graphs. Combining the two previous theorems shows that the edge polytope of bipartite graphs coincides with the stable set polytope. We have even more:

Theorem 3.4 *ESTAB(G) = STAB(G) if and only if G is bipartite.*

Prove as exercise: If G contains an odd cycle then ESTAB$(G) \neq$ STAB(G).

Example. The $K_3 = C_3$ is the smallest graph G with ESTAB$(G) \neq$ STAB(G) due to the following reason. The vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ belongs to ESTAB(C_3) since it obviously satisfies all constraints. However, this vector is not in STAB(C_3) since it cannot be obtained as a convex combination of the incidence vectors of the stable sets of the C_3:

$$\lambda_0 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

would imply $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$, a contradiction to $\sum \lambda_i = 1$.

3.4 The clique polytope QSTAB(G)

In order to find, for non-bipartite graphs $G = (V, E)$, facets of the stable set polytope, we may look for a common generalization of the inequalities $x_i \le 1 \; \forall i \in V$ and $x_i + x_j \le 1 \; \forall ij \in E$ building the edge polytope. Obviously, **clique constraints**

$$\sum_{i \in Q} x_i \le 1, \; Q \subseteq G \text{ clique} \tag{3.13}$$

associated with cliques of arbitrary size generalize node and edge constraints. Clique constraints are clearly valid for STAB(G) since the incidence vector χ^S of any stable set S of G satisfies these inequalities due to $|S \cap Q| \le 1$. We infer

$$\text{STAB}(G) = \text{conv}\{x \in \{0,1\}^{|G|} : x \text{ satisfies } (3.13)\}$$

and thus

$$\max \mathbb{1}^T x, \; x(Q) \le 1 \; \forall \text{cliques } Q \subseteq G, \; x \in \mathbb{R}_+^{|G|} \tag{3.14}$$

is an LP-relaxation of the stable set problem (3.6). The solution space of (3.14) is called the **clique polytope** (or fractional stable set polytope)

$$\text{QSTAB}(G) = \{x \in \mathbb{R}_+^{|G|} : x \text{ satisfies } (3.13)\}.$$

We have clearly STAB(G) $\subseteq$ QSTAB(G) $\subseteq$ ESTAB(G) and are interested in those graphs G for which QSTAB(G) contains all facets of STAB(G).

First note that clique constraints associated with *non-maximal* cliques Q of a graph G are always redundant: in this case, G admits a clique Q^* with $Q \subset Q^*$ and the inequality $x(Q) \le 1$ is dominated by $x(Q^*) \le 1$. Hence we need to consider clique constraints associated with *maximal* cliques of G only in order to obtain *facet-defining* inequalities for STAB(G). Padberg showed even more:

Lemma 3.5 (Padberg 1974 [50]) *A clique constraint $x(Q) \le 1$ defines a facet of STAB(G) if and only if Q is a maximal clique of G.*

Proof. In order to show that $x(Q) \le 1$ defines a facet whenever Q is a maximal clique of $G = (V, E)$, we construct $|V|$ many roots of the clique constraint having linearly independent incidence vectors. Clearly, $\chi^{\{i\}}$ satisfies the constraint at equality for all nodes $i \in Q$. Due to the maximality of Q, there exists, for all nodes $j \in G - Q$, a non-neighbor $i \in Q$ and $\chi^{\{i,j\}}$ satisfies the constraint at equality, too.

Let A be the matrix having the incidence vectors of these $|V|$ stable sets as rows and the nodes of G as columns s.t. the incidence vectors $\chi^{\{i,j\}}$ are listed before the incidence vectors $\chi^{\{i\}}$ and that the nodes from $G - Q$ are listed before the nodes from Q:

$$A = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right)$$

Due to its block structure, A is obviously non-singular. Hence, the incidence vectors of the constructed roots are linearly independent and $x(Q) \leq 1$ is indeed a facet of STAB(G).

Conversely, if G admits a clique Q^* with $Q \subset Q^*$, then there is, for each node $j \in Q^* - Q$, no stable set S containing j which satisfies $x(Q) \leq 1$ at equality. Hence the matrix having the incidence vectors of all roots of $x(Q) \leq 1$ as rows admits a 0-column for each $j \in Q^* - Q$ and is, therefore, singular. Consequently, the clique constraint associated with a non-maximal clique Q cannot define a facet of STAB(G). □

The lemma shows that the maximal clique constraints are *necessary* for describing the stable set polytope of any graph. We are now interested in these graphs G for which they are also *sufficient*, i.e., for which STAB(G) and QSTAB(G) coincide.

The Theorem 3.2 of Hoffman & Kruskal implies STAB(G) = QSTAB(G) for all graphs G where the matrix having the incidence vectors of all maximal cliques as rows is totally unimodular. Graphs G with totally unimodular maximal clique/node incidence matrix are called **unimodular**. All bipartite graphs are unimodular by Theorem 3.3.
Show as exercise: Line graphs of bipartite graphs are unimodular, too.

We look for the class of graphs G that is characterized by STAB(G) = QSTAB(G).

3.4.1 A polyhedral characterization of perfect graphs

We give a polyhedral characterization of perfect graphs in a more general context. Consider a weighted graph $(G, c) = (V, E, c)$ with $c : V \to \mathbb{Z}_+$ and let $c_i = c(i)$ for all $i \in V$. To find the *weighted stability number*

$$\alpha(G, c) = \max\{\sum_{i \in S} c_i : S \subseteq G \text{ stable}\}$$

is obviously equal to

$$\max c^T x, x \in \text{STAB}(G)$$

and we have particularly $\alpha(G) = \alpha(G, \mathbb{1})$. We may generalize the stable set problem to "find a stable set of maximum cardinality or maximum weight". In order to establish for which graphs G the stable set problem can be solved via the LP

$$\max c^T x, x \in \text{QSTAB}(G)$$

we make use of LP-duality:

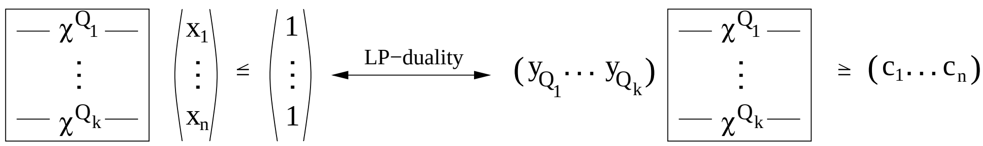
Theorem 3.6 (Duality Theorem of Linear Programming, J. von Neumann 1947 [48]) *The linear programs*

$$\begin{array}{rrrl}
(P) & \max & c^T x & \\
& & Ax & \leq b \\
& & x & \geq 0
\end{array}
\quad \textit{and} \quad
\begin{array}{rrrl}
(D) & \min & y^T b & \\
& & y^T A & \geq c^T \\
& & y & \geq 0
\end{array}$$

have optimal solutions with equal objective function value if and only if they admit feasible solutions.

The following chain of inequalities and equations is typical for IP/LP approaches to combinatorial problems:

$$\begin{array}{rcl}
\alpha(G, c) & = & \max\{\sum_{i \in S} c_i : S \subseteq G \text{ stable}\} \\
& = & \max\{c^T x : x \in \text{STAB}(G)\} \\
& = & \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \geq 0,\ x \in \{0,1\}^{|G|}\} \\
& \leq & \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \geq 0\} \\
& = & \max\{c^T x : Ax \leq \mathbb{1},\ x \geq 0\} \text{ with } A \text{ clique/node incidence matrix}
\end{array}$$



$$\begin{aligned}
&= \min\{y^T \mathbb{1} : y^T A \geq c^T,\ y \geq 0\} \text{ by LP} - \text{duality} \\
&= \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i\ \forall i \in G,\ y_Q \geq 0\ \forall \text{cliques } Q \subseteq G\} \\
&\leq \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i\ \forall i \in G,\ y_Q \geq 0,\ y_Q \in \mathbb{Z}_+ \\
&\quad \forall \text{cliques } Q \subseteq G\} \\
&= \overline{\chi}(G, c)
\end{aligned}$$

The inequalities come from dropping or adding integrality constraints, one of the equations is implied by LP-duality. The last program can be interpreted as an IP formulation of the *weighted clique cover problem*.

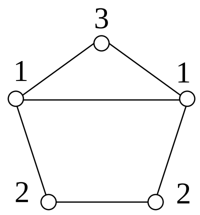
It follows from the PGT that equality holds throughout the whole chain for all 0/1-vectors c iff G is perfect. The answer to the question for which graphs equality holds for *all* possible node weightings c results in a polyhedral characterization of perfect graphs and a weighted version of the PGT involving the weighted clique number $\omega(G, c) = \alpha(\overline{G}, c)$ and the weighted chromatic number $\chi(G, c) = \overline{\chi}(\overline{G}, c)$.

Theorem 3.7 (Weighted PGT [37]) *For any graph $G = (V, E)$, the following conditions are equivalent:*

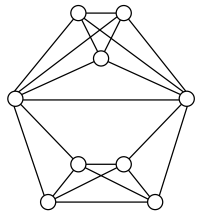
(i) $\omega(G') = \chi(G')$ *for each induced subgraph* $G' \subseteq G$.
(ii) $\omega(G, c) = \chi(G, c)$ *for each weighting* $c : V \to \mathbb{Z}_+$.
(iii) STAB(G) = QSTAB(G).
(iv) $\overline{G}$ *satisfies* (i).
(v) $\overline{G}$ *satisfies* (ii).
(vi) $\overline{G}$ *satisfies* (iii).

Remark. The original proof of
(i) $\Rightarrow$ (ii) and (i) $\Leftrightarrow$ (iv) is due to Lovász 1972 [44],
(ii) $\Rightarrow$ (v) $\Rightarrow$ (iv) is due to Fulkerson 1971 [32],
(i) $\Leftrightarrow$ (iii) is independently due to Chvátal 1975 [14] and Padberg 1974 [50].

Proof of (i) $\Rightarrow$ (ii) by Lovász [44] via the Replication Lemma. Replicating a node x of a graph G means to add a new node x' to G which is adjacent to x and all neighbors of x. Lovász established that replicating an arbitrary node of a perfect graph yields a perfect graph again (Replication Lemma). Thus we can construct, from any weighted perfect graph (G, c), an unweighted perfect graph G^c by replicating each node i of G precisely c_i times. The assertion follows due to $\omega(G^c) = \omega(G, c)$ and $\chi(G^c) = \chi(G, c)$. $\diamond$



(G,c)



G^c

Proof of (ii) ⇒ (iii) by Grötschel, Lovász & Schrijver [37]. Let $G = (V, E)$ and $y \in \mathbb{Q}^{|V|}$ be an arbitrary vector from QSTAB(G). We show $y \in$ STAB(G) (in order to establish QSTAB(G) $\subseteq$ STAB(G)). Let $q \in \mathbb{Z}$ be the least common denominator of the entries in y. Then $qy \in \mathbb{Z}_+^{|V|}$, $y \in$ QSTAB(G), and the clique constraint

$$\sum_{i \in Q} x_i \leq 1$$

associated with any clique Q of G imply

$$\omega(G, qy) = \max_Q \sum_{i \in Q} qy_i \leq q.$$

By (ii), we have $\chi(G, qy) \leq q$ and there exists a family $S_1, \ldots, S_q$ of stable sets s.t. each node i of G is contained in exactly qy_i of them. In other words,

$$qy = \sum_{i \leq q} \chi^{S_j} \text{ resp. } y = \frac{1}{q} \sum_{i \leq q} \chi^{S_j}$$

shows that y can be represented as a convex combination of points in STAB(G) and $y \in$ STAB(G) proves finally QSTAB(G) $\subseteq$ STAB(G). ◇

Proof of (iii) ⇒ (iv) by Grötschel, Lovász & Schrijver [37]. If STAB(G) is determined by nonnegativity and clique constraints only, then the same is true for every induced subgraph G' of G. Hence it suffices to show that G itself can be partitioned into $\alpha(G)$ cliques (in order to show that every induced subgraph $\overline{G}'$ of $\overline{G}$ can be partitioned into $\omega(\overline{G}')$ stable sets). We use induction on $|G|$.

Let F be the face of STAB(G) spanned by all stable sets of size $\alpha(G)$ (i.e., F contains all convex combinations of the incidence vectors of the maximum stable sets of G). There exists a clique facet of STAB(G) containing F, i.e., a facet of the form $x(Q) \leq 1$ with Q clique (otherwise F would be the intersection of some nonnegativity facets, a contradiction to $\chi^{\emptyset} = (0, \ldots, 0) \notin F$ by construction).

But this means that, for each maximum stable set S of G, we have $|S \cap Q| = \chi^S Q = 1$ and, therefore, $\alpha(G - Q) = \alpha(G) - 1$. By the induction hypothesis, $G - Q$ can be partitioned into $\alpha(G - Q)$ cliques. Adding Q to this family of cliques, we obtain the studied clique cover of G using $\alpha(G)$ cliques. $\diamond$

The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i) follow by interchanging the roles of G and $\overline{G}$. $\square$

The latter theorem provides the link between perfect graphs and integer programming ('a graph is perfect iff certain LP's have integral solution values') and polyhedral theory ('a graph is perfect iff certain polyhedra are identical').

In particular, we obtain an answer to the question for which graphs the stable set polytope and the clique polytope coincide:

Corollary 3.8 *$STAB(G) = QSTAB(G)$ if and only if G is perfect.*

3.4.2 The Theta-Body

One is attempted to expect that the optimization problem for the LP-relaxation QSTAB(G) of STAB(G) is polynomially solvable. This expectation is even more justified since QSTAB(G) has nice an 'easily recognizable' facets. However, Grötschel, Lovász & Schrijver established the following:

Theorem 3.9 (Grötschel, Lovász & Schrijver 1981 [36]) *The optimization problem for* QSTAB(G) *is NP-hard in general.*

Sketch of the proof. The optimization problem for any polytope $P \in \mathbb{R}^n$ is polynomially equivalent to the optimization problem for its antiblocker

$$\text{abl}(P) = \{x \in \mathbb{R}^n_+ : x^T y \leq 1 \ \forall y \in P\}.$$

The antiblocker of QSTAB(G) is the polytope STAB($\overline{G}$) (and the antiblocker of STAB(G) is QSTAB($\overline{G}$)). Thus the optimization problem for QSTAB(G) is polynomially equivalent to the stable set problem for $\overline{G}$, which is NP-hard for general graphs by Theorem 1.9 due to Karp. $\square$

For the class of perfect graphs, though, the optimization problem for QSTAB(G) (and, therefore, for STAB(G)) can be solved in polynomial time - albeit an involved detour: a geometric representation of graphs introduced by Lovász [46] in 1979.

Let $G = (V, E)$ be a graph. An **orthonormal representation (ONR)** of G is a sequence $(u_i : i \in V)$ of $|V|$ vectors $u_i \in \mathbb{R}^N$, where N is some positive integer, s.t.

- $||u_i|| = 1$ for all $i \in V$ and
- $u_i^T u_j = 0$ for all $ij \notin E$.

Trivially, every graph has an ONR: just take all the vectors u_i mutually orthogonal in $\mathbb{R}^{|V|}$. Figure 3.3 shows a less trivial ONR of the C_5 in $\mathbb{R}^3$ constructed as follows. Consider an umbrella with five ribs of unit length (representing the nodes of the C_5) and open the umbrella in such a way that non-adjacent ribs become orthogonal. Clearly, this can be achieved in $\mathbb{R}^3$ and gives an ONR of the C_5.

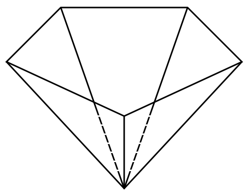


Figure 3.3: An ONR of the C_5 in $\mathbb{R}^3$

For any ONR $(u_i : i \in V)$, $u_i \in \mathbb{R}^N$ of G and any additional vector $c \in \mathbb{R}^N$ of unit length, the **orthonormal representation constraint**

$$\sum_{i \in V} (c^T u_i)^2 x_i \leq 1 \tag{3.15}$$

is valid for STAB(G) due to the following reason. For any stable set S of G, the vectors $u_i, i \in S$ are mutually orthogonal by construction and, therefore, $\sum_{i \in S} (c^T u_i)^2 \leq 1$ follows. We obtain

$$\sum_{i \in V} (c^T u_i)^2 \chi_i^S = \sum_{i \in S} (c^T u_i)^2$$

for the incidence vector of any stable set S of G yielding the validity of the ONR constraints for STAB(G).

Moreover, taking an orthonormal basis $B = \{e_1, \ldots, e_{|V|}\}$ of $\mathbb{R}^{|V|}$ and a clique Q of G, we obtain an ONR by setting $c = u_i = e_1$ for all $i \in Q$ and assigning different vectors of $B - \{e_1\}$ to all the remaining nodes $j \in G - Q$. Then the corresponding ONR constraint is just the clique constraint associated with Q (by $c^T u_i = 1$ for $i \in Q$ and $c^T u_j = 0$ otherwise). Hence, every clique constraint is a special ONR constraint.

For any graph $G = (V, E)$, the set

$$\mathrm{TH}(G) = \{x \in \mathbb{R}_+^V : x \text{ satisfies } (3.15)\}$$

is the intersection of infinitely many halfspaces (since G admits infinitely many ONR's), so $\mathrm{TH}(G)$ is a convex set but no polytope in general. The above remarks imply

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G) \tag{3.16}$$

and Corollary 3.8 shows that all three coincide if and only if G is perfect. Even more, Grötschel, Lovász & Schrijver established the following:

Theorem 3.10 (Grötschel, Lovász & Schrijver 1988 [37]) *If an inequality defines a facet of* $\mathrm{TH}(G)$*, then it is a positive multiple of a nonnegativity or a clique constraint.*

This result implies a further polyhedral characterization of perfect graphs:

Theorem 3.11 (Grötschel, Lovász & Schrijver 1988 [37])
The following conditions are equivalent:

(0) G *is perfect.*

(i) $\mathrm{TH}(G)$ *is a polytope.*

(ii) $\mathrm{STAB}(G) = \mathrm{TH}(G)$.

(iii) $\mathrm{TH}(G) = \mathrm{QSTAB}(G)$.

Proof. We show that (0) is equivalent to all other three conditions. If G is perfect, then $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ follows from Corollary 3.8. Thus, the inclusion (3.16) shows all three assertions (i), (ii), and (iii).

Conversely, suppose one of the three assertions (i), (ii), and (iii) is true. Then $\mathrm{TH}(G)$ is in particular a polytope and is, therefore, the solution set of all inequalities determining a facet of $\mathrm{TH}(G)$. By Theorem 3.10, all these inequalities are either nonnegativity or clique constraints yielding

$\mathrm{TH}(G) = \mathrm{QSTAB}(G)$. The set $\mathrm{TH}(G)$ equals the antiblocker $\mathrm{abl}(\mathrm{TH}(\overline{G}))$ by Grötschel, Lovász & Schrijver [36]. Thus, $\mathrm{TH}(\overline{G})$ is a polytope, too. Applying the same argumentation as above, we obtain $\mathrm{TH}(\overline{G}) = \mathrm{QSTAB}(\overline{G})$. But now,

$$\mathrm{STAB}(G) = \mathrm{abl}(\mathrm{QSTAB}(\overline{G})) = \mathrm{abl}(\mathrm{TH}(\overline{G})) = \mathrm{TH}(G) = \mathrm{QSTAB}(G)$$

follows and thus G is perfect due to Corollary 3.8. □

This result is particularly remarkable since it states that a graph is perfect if and only if a certain convex set is a polytope.

The key property of $\mathrm{TH}(G)$ for linear programming was established by Grötschel, Lovász & Schrijver again:

Theorem 3.12 (Grötschel, Lovász & Schrijver 1981 [36]) *If $c \in \mathbb{R}_+^V$ is a vector of node weights, the optimization problem (with infinitely many linear constraints)*

$$\max c^T x, \ \ x \in \mathrm{TH}(G)$$

can be solved in polynomial time for any graph $G = (V, E)$.

This deep result rests on the fact that the value

$$\vartheta(G, c) = \max\{c^T x : x \in \mathrm{TH}(G)\}$$

can be characterized in many equivalent ways, e.g., as the

- optimum value of a semidefinite program,
- largest eigenvalue of a certain set of symmetric matrices,
- maximum value of some function involving ONR's

(see Grötschel, Lovász & Schrijver [37] for the details). Theorem 3.11 (resp. the inclusion (3.16) together with Corollary 3.8) finally imply

$$\alpha(G, c) = \vartheta(G, c)$$

for all perfect graphs and, hence, Theorem 3.12 shows:

Corollary 3.13 (Grötschel, Lovász & Schrijver 1981 [36]) *The stable set problem can be solved in polynomial time for perfect graphs.*

Therefore, the

- clique cover number $\overline{\chi}(G, c) = \alpha(G, c)$,
- chromatic number $\chi(G, c) = \overline{\chi}(\overline{G}, c)$,
- clique number $\omega(G, c) = \alpha(\overline{G}, c)$

can be computed in polynomial time for perfect graphs G and any node weighting c, too.

3.5 The rank polytope RSTAB(G)

In order to find, for imperfect graphs, facets of the stable set polytope, we may look for a generalization of the clique constraints. Obviously, **rank constraints**

$$\sum_{i \in G'} x_i \le \alpha(G'), \; G' \subseteq G \tag{3.17}$$

associated with *arbitrary* induced subgraphs G' of G generalize clique constraints. Rank constraints are clearly valid for STAB(G). (Note that cliques are precisely those induced subgraphs G' with $\alpha(G') = 1$.) We infer

$$\mathrm{STAB}(G) = \mathrm{conv}\{x \in \{0,1\}^{|G|} : x \text{ satisfies } (3.17)\}$$

and thus

$$\max \mathbb{1}^T x, \; x(G', \mathbb{1}) \le \alpha(G') \; \forall G' \subseteq G, \; x \in \mathbb{R}_+^{|G|} \tag{3.18}$$

is an LP-relaxation of the stable set problem (3.6). The solution space of (3.18) is called the **rank polytope**

$$\mathrm{RSTAB}(G) = \{x \in \mathbb{R}_+^{|G|} : x \text{ satisfies } (3.17)\}.$$

We have clearly STAB(G) $\subseteq$ RSTAB(G) $\subseteq$ QSTAB(G) and are interested in those graphs G for which RSTAB(G) contains all facets of STAB(G). We call a graph G **rank perfect** [60] if RSTAB(G) = STAB(G).

There is no characterization of rank-perfect graphs known yet. We are, therefore, interested in identifying certain classes of rank-perfect graphs. Obviously, all perfect graphs are rank-perfect by definition. In the sequel, we consider further classes of rank-perfect graphs obtained by allowing certain *kinds* of rank constraints as only nontrivial facets of the stable set polytope, see [60]. Perfect graphs are, e.g., those rank-perfect graphs where we restrict the rank constraints to clique constraints.

3.5.1 Near-perfect graphs

The rank constraint

$$x(G, \mathbb{1}) \le \alpha(G) \tag{3.19}$$

associated with the graph G itself is called the **full rank constraint**. Shepherd [57] termed a graph G **near-perfect** if its stable set polytope is given by clique constraints and the full rank constraint, i.e., if

$$\mathrm{STAB}(G) = \mathrm{QSTAB}(G) \cap \{x \in \mathbb{R}^{|G|} : x(G, \mathbb{1}) \le \alpha(G)\}$$

holds. Obviously, perfect graphs are all near-perfect. If an imperfect graph G is near-perfect, then the full rank constraint is needed as *facet* and we say that G *produces* the full rank facet. This is true for, e.g., all minimally imperfect graphs.

Theorem 3.14 (Padberg 1974 [50]) *G is minimally imperfect if and only if* $\text{QSTAB}(G)$ *has exactly one fractional extreme point (namely, $\frac{1}{\omega(G)}\mathbb{1}$ which is adjacent to the $|G|$ integer extreme points coming from the maximum stable sets of G) and* $\text{STAB}(G) = \text{QSTAB}(G) \cap \{x \in \mathbb{R}^{|G|} : x(G, \mathbb{1}) \le \alpha(G)\}$.

Hence, minimally imperfect graphs are those imperfect graphs G where $\text{QSTAB}(G)$ and $\text{STAB}(G)$ are ‘as close as possible‘. In particular:

Corollary 3.15 *Minimal imperfect graphs are near-perfect.*

Show as exercise: Odd holes and odd antiholes produce the full rank facet.

Shepherd found a further polyhedral characterization of minimally imperfect graphs in terms of near-perfection:

Theorem 3.16 (Shepherd 1994 [57]) *An imperfect graph G is minimally imperfect if and only if both G and $\overline{G}$ are near-perfect.*

That means, the part of the class of near-perfect graphs which is closed under complementation consists exactly in all perfect and all minimally imperfect graphs.

Besides perfect and minimally imperfect graphs, no other class is known so far to belong (completely) to the class of near-perfect graphs. Shepherd found a characterization of near-perfect graphs with stability number two.

Theorem 3.17 (Shepherd 1994 [57]) *A graph G with $\alpha(G) = 2$ is near-perfect if and only if the neighborhood of every node of G induces a perfect graph.*

3.5.2 T-perfect and h-perfect graphs

The rank constraint

$$x(C_{2k+1}, \mathbb{1}) \le k \tag{3.20}$$

associated with an odd hole $C_{2k+1} \subseteq G$ is called **odd hole constraint**. Denote by

$$\text{CSTAB}(G) = \{x \in \mathbb{R}_+^{|G|} : x(C_{2k+1}, \mathbb{1}) \le k \ \forall C_{2k+1} \subseteq G\}$$

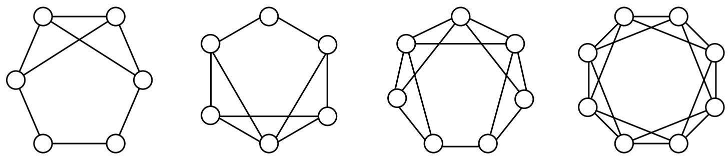


Figure 3.4: Examples of near-perfect graphs

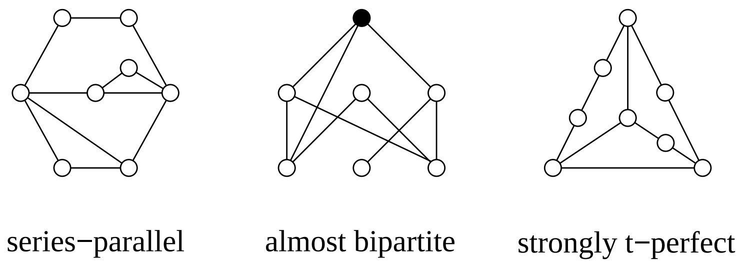
the cycle polytope of G. Chvátal [14] termed a graph G **t-perfect** if its stable set polytope is given by rank constraints associated with nodes, edges, triangles, or odd holes, i.e., if

$$\text{STAB}(G) = \text{ESTAB}(G) \cap \text{CSTAB}(G)$$

holds. (Note that "t" stands for "trou", the French word for hole, and that every C_{2k+1} with $k \geq 1$ is here considered to be a hole.)

Examples of t-perfect graphs are:

- Bipartite graphs and odd holes.
- *Series-parallel graphs* by Boulala & Uhry [8] in 1979
(graphs obtained from disjoint cycle-free subgraphs by repeated application of the following two operations: adding a new edge parallel to an existing edge and subdividing edges, i.e., replacing edges by a path).
- *Almost bipartite graphs* due to Fonlupt & Uhry [31] in 1982
(graphs having a node the deletion of which leaves the graph bipartite)
- *Strongly t-perfect graphs* by Gerards & Schrijver [35] in 1986
(graphs having no subgraph obtained from subdividing edges of a K_4 such that all four cycles corresponding to the triangles of the K_4 are odd)



However, even small graphs as the K_4 are *not* t-perfect. This motivated Grötschel, Lovász & Schrijver [37] in 1988 to define a natural generalization of t-perfect graphs. The class of **h-perfect graphs** (from hole-perfect) contains all graphs where rank constraints associated with cliques of arbitrary size and odd holes suffice to describe the associated stable set polytopes:

$$\text{STAB}(G) = \text{QSTAB}(G) \cap \text{CSTAB}(G)$$

One class of nontrivial h-perfect graphs (that are neither perfect, nor t-perfect, nor combinations of these) is the class of (P_5,diamond)-free graphs due to Arbib & Mosca [5] in 2002.

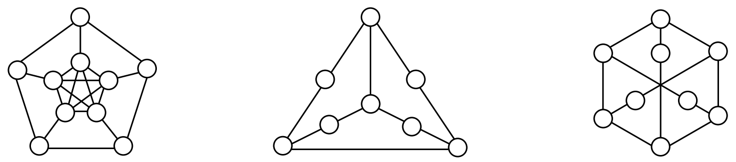


Figure 3.5: Examples of h-perfect but not t-perfect graphs

Grötschel, Lovász & Schrijver [37] proved that there is a polynomial time algorithm to solve the separation problem for odd hole constraints. So we may take the nonnegativity constraints, the odd hole constraints (3.20), and the orthonormal representation constraints (3.15) in order to obtain a convex set containing STAB(G) for which the optimization problem can be solved in polynomial time by Theorem 3.12. For any h-perfect graph G,

$$\text{STAB}(G) = \text{TH}(G) \cap \text{CSTAB}(G)$$

holds and thus the stable set problem can be solved in polynomial time.

3.5.3 Line graphs

Line graphs (of general graphs) are a further class of rank-perfect graphs due to a result of Edmonds & Pulleyblank [29] in 1974. They characterized the system of facet-defining inequalities for the *matching polytope*. Since matchings of the original graph correspond to stable sets of the line graph, their result implies a description of the stable set polytope of line graphs.

Remark. Note that line graphs are a "natural" graph class which is proved to consist of rank-perfect graphs only (while near-perfect, t-perfect, and h-perfect graphs are rank-perfect by definition).

The **matching polytope** $\mathrm{M}(G)$ of a graph $G = (V, E)$ is defined as the convex hull of the incidence vectors of all matchings of G:

$$\mathrm{M}(G) := \mathrm{conv}\{\chi^M \in \mathbb{R}_+^{|E|} : M \subseteq E \text{ matching}\}$$

Edmods described in 1965 the matching polytope as follows:

Theorem 3.18 (Edmods [28]) $\mathrm{M}(G)$ *contains all* $x \in \mathbb{R}_+^{|E|}$ *satisfying*

(i) $x(\delta(v)) \leq 1\ \forall v \in V$ *where* $\delta(v) = \{e \in E : v \in e\}$, *and*

(ii) $x(E(V')) \leq \frac{|V'|-1}{2}\ \forall V' \in \mathcal{O}$ *where* $\mathcal{O} = \{V' \subseteq V : |V'| \geq 3, \text{odd}\}$.

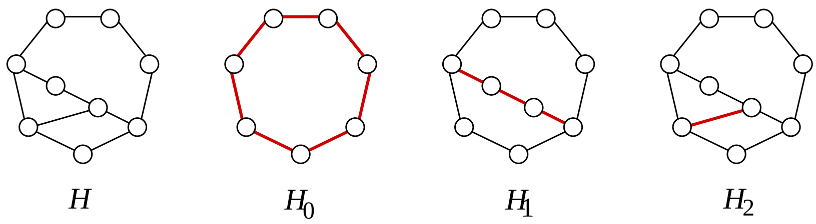
The condition (i) is always necessary to describe $\mathrm{M}(G)$. (*Show as exercise: (i) corresponds to maximal clique constraints of the stable set polytope!*)

Later on, Edmonds & Pulleyblank [29] figured out which 'odd sets' from $\mathcal{O}$ give rise to *facets* of $\mathrm{M}(G)$. To present the result, we need the following notations.

A graph H is called **hypomatchable** if, for all nodes v of H, the subgraph $H - v$ admits a perfect matching, i.e. a matching meeting all nodes. Due to a result by Lovász [45] in 1972, H is hypomatchable iff there is a sequence $H_0, H_1, \ldots, H_k = H$ of graphs such that

- H_0 is a chordless odd cycle and for $1 \leq i \leq k$,
- H_i is obtained from H_{i-1} by adding an odd path E_i that joins two (not necessarily distinct) nodes of H_{i-1} and has all internal nodes outside H_{i-1}.

The odd paths $E_i = H_i - H_{i-1}$ are called *ears* for $1 \leq i \leq k$ and the sequence $H_0, H_1, \ldots, H_k = H$ an *ear decomposition* of H.



Hypomatchable graphs have an odd number of nodes, are non-bipartite and connected, but not necessarily 2-connected (a graph is 2-connected if it is still connected after removing an arbitrary node).

By definition, an ear may be an odd cycle attached to a single node of the previous graph. However, these degenerated ears can be avoided if H is 2-connected: Cornuéjols & Pulleyblank [25] proved in 1983 that every 2-connected hypomatchable graph H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$ with H_i 2-connected for $0 \leq i \leq k$.

Theorem 3.19 (Edmonds & Pulleyblank 1974 [29]) *If $H = (V, E)$ is a 2-connected hypomatchable graph then*

$$x(E) \leq \frac{|V| - 1}{2}$$

is a facet of $\mathrm{M}(H)$.

Proof. We show that H admits $|E|$ matchings of maximum size $\frac{|V|-1}{2}$ the incidence vectors of which are linearly independent.

Consider an ear decomposition $H_0, H_1, \ldots, H_k = H$ s.t. each H_i is 2-connected. By induction each H_i has $|E(H_i)|$ maximum matchings whose incidence vectors are linearly independent: The assertion is obviously satisfied by the odd cycle H_0. Suppose it is true for H_{i-1} and consider H_i.

Let v and v' be the endnodes of the ear E_i (recall: $v \neq v'$ by H_i 2-connected). By the induction hypothesis, there are $|E(H_{i-1})|$ independent maximum matchings of H_{i-1}.

Each can be extended to a maximum matching of H_i by adding the 2nd, 4th, ... edges of E_i.

Then $|E(E_i)| - 1$ additional matchings can be obtained by considering each internal node w of E_i, if one constructs a matching of E_i which does not meet w and $w' \in \{v, v'\}$ s.t. the (w, w')-path in E_i has odd length.

We extend this matching of E_i to a maximum matching of H_i by adding a matching of H_{i-1} which does not meet only the endnode of E_i distinct from w'. This is the only matching constructed so far not meeting w, so it is independent of all others.

We get the last matching by starting with the 1st, 3rd, ... edges of E_i and combining it with a perfect matching of $H_{i-1} - v$ from which the edge incident to v' has been removed (if the ear E_i has length one, this is the only new matching to construct). The last matching contains one edge of H_{i-1} less than the others, so it is independent of all others. □

Furthermore, Edmonds & Pulleyblank [29] characterized which hypomatchable *subgraphs* of G produce facets of $\mathrm{M}(G)$.

Theorem 3.20 (Edmonds & Pulleyblank 1974 [29]) *Let $H \in \mathcal{O}$ be s.t. the inequality (ii) associated with H is not of form (i) for some $v \in V$. Then*

$$x(E(H)) \le \frac{|V(H)|-1}{2}$$

is a facet of $\mathrm{M}(G)$ iff H induces a 2-connected hypomatchable subgraph of G.

Theorem 3.20 implies that the stable set polytopes of line graphs $L(G)$ admit nonnegativity constraints, clique constraints, and rank constraints

$$x(L(H)) \le \frac{|V(H)|-1}{2} \tag{3.21}$$

associated with the line graphs of 2-connected hypomatchable induced subgraphs $H \subseteq G$ only. Consequently:

Corollary 3.21 *Line graphs are rank-perfect.*

We finally obtained the following inclusion relations of the graph classes discussed within this chapter:

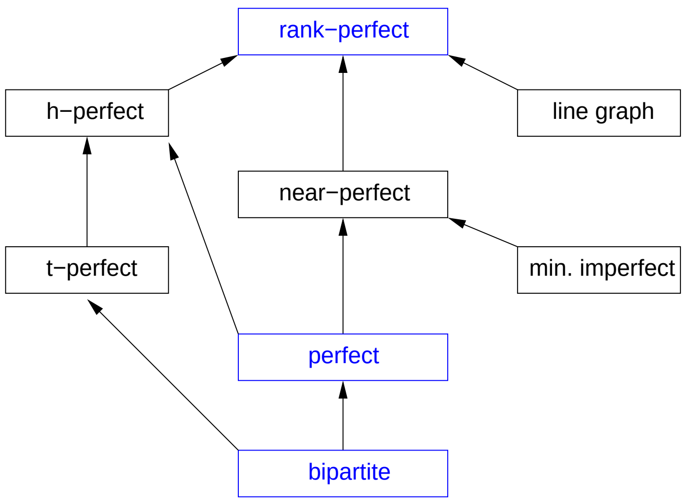


Figure 3.6: Inclusion relations.

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