Pricing and hedging options: an introduction from an optimization perspective

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Basics of Option Pricing
  Single-period
  Discrete time models

American Contingent Claims
  American Contingent Claims
  Background on Pricing ACCs

Financial Market
  The Scenario Tree
  Market Instruments and Portfolios

Exact Linear Relaxation
  Statement of Theorem
  Sketch of Proof

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The market

- Two dates: $t = 0, t = 1$
- Finite sample space
  \[ \Omega = \{ \omega_1, \omega_2, \ldots, \omega_m \} \]
- $n$ securities $S^1, S^2, \ldots, S^n$
- $S_0 = (S^1_0, S^2_0, \ldots, S^n_0)$ is the $n$-vector of prices at time $t = 0$
- $S_1 = (S^1_1|S^2_1|\ldots|S^n_1)$ is a $mxn$ matrix.
Example 1

\[
\begin{array}{c}
10 \\
15 \\
7.5
\end{array}
\]

\[
\begin{array}{c}
t=0 \\
t=1
\end{array}
\]
Portfolios of securities

- At time $t = 0$ it is possible to take any position ("long" or "short") on the $n$ securities.
- Let $\mathbf{x}$ be a portfolio of securities

\[
\mathbf{x} = \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\]

The cost (at $t = 0$) of $\mathbf{x}$ is $\mathbf{S}_0 \mathbf{x}$ (a scalar)
The payoff (at $t = 1$) of $\mathbf{x}$ is $\mathbf{S}_1 \mathbf{x}$ ($m$-vector)
B-Arbitrages

- Let us find the minimum cost portfolio with positive payoff
- The Primal Problem (P)

\[
\min_x S_0 x \\
S_1 x \geq 0
\]

- (P) is feasible, hence it is either bounded or unbounded
- (P) is unbounded iff it is possible to make a profit in \( t = 0 \) (\( S_0 x < 0 \)), with no future liabilities (\( S_1 x \geq 0 \)). This is an arbitrage (of type B).
- (P) bounded \( \iff \) No B-arbitrages; B-arbitrages are not consistent with economic equilibrium
The Dual problem

- The Dual of (P) is the following LP (D)

\[
\begin{align*}
\max_y y \cdot 0 \\
S_1' y &= S_0 \\
y \geq 0
\end{align*}
\]

- (D) feasible \iff (P) bounded \iff No B-arbitrages
There is a second type of arbitrage: a free lottery ticket.

Portfolio \( x \) is an **A-arbitrage** if

\[
S_0 x = 0 \\
S_1 x \geq 0 \\
S_1 x(\omega_i) > 0 \quad \text{for some } \omega_i \in \Omega
\]

A-arbitrages are not consistent with economic equilibrium.

A model is "arbitrage-free" if there are neither A nor B arbitrages.
Arbitrage-free models

- A market model is arbitrage-free ⇔ There is a strictly positive solution to

\[ S_1' y = S_0 \]
\[ y \geq 0 \]

- Define \( B_0 = \sum_{i=1}^{m} y_i \); then \( q = y / B_0 \) is a probability on \( \Omega \)
- \( q \) is called ”risk-neutral” because \( S_0 = B_0 \mathbb{E}^q S_1 \)
- No arbitrage ⇔ \( \exists q \) risk-neutral
Example 2
Pricing contingent claims

- Assume there are no arbitrages
- A contingent claim \( b \) is a random variable on \( \Omega \).
- A portfolio \( x \) "replicates" \( b \) if

\[
S_1x = b
\]

- A claim is **attainable** if it admits a replicating portfolio
- The no-arbitrage price of an attainable claim \( b \) is

\[
S_0x = x \cdot B_0E^qS_1 = B_0E^qS_1x = B_0E^qb
\]
Complete markets

- A market is "complete" if all claims are attainable.
- Market is complete ⇔ \( \text{lin}\{S_1^1, \ldots, S_1^n\} = \mathbb{R}^m \)
- That is \( n \geq m \) and
  \[
  \text{rank}(S_1) = m
  \]
- The Dual Problem (D) has a unique solution.
- Completeness (and No-arbitrages) ⇔ \( \exists!q \)
Incomplete markets

- Suppose a claim \( b \) is not attainable
- We can determine the minimum price \( V^+ \) for a "super-replicating" strategy
- This is called the "writer’s” problem: if one buys the claim at any price greater than \( V^+ \) there is an arbitrage opportunity for the writer of the option
- Analogously, the ”buyer’s” problem finds the maximum price \( V^- \) for a sub-replicating strategy
The writer’s problem

Consider

\[ V^+ = \min_x S_0 x \]
\[ S_1 x \geq b \]

Its dual is

\[ \max_y b'y \]
\[ S_1'y = S_0 \]
\[ y \geq 0 \]
Therefore

\[ V^+ = \max_{q \in Q} B_0 \mathbb{E}^q b \]

where \( Q \) is the set of risk-neutral measures (in fact, closure of the set of risk-neutral measures equivalent to \( p \)).
Arbitrage-free prices of non-attainable claims

- The analogous "buyer’s problem" yields to

\[ V^- = \min_{q \in \mathcal{Q}} B_0 \mathbb{E}^q b \]

- Any price \( V \), such that \( V^- \leq V \leq V^+ \) is an arbitrage free price (all inequalities are strict if \( V^- < V^+ \))

- A claim is attainable iff it has a unique arbitrage-free price
Market frictions

- Suppose there are different bid-ask prices
- The primal problem (P) becomes

\[
\begin{align*}
\min_{x^a, x^b} & \quad S_0^a x^a - S_0^b x^a \\
S_1^a x^a - S_1^b x^b & \geq 0 \\
x^a & \geq 0 \\
x^b & \geq 0
\end{align*}
\]
Market frictions

The dual problem is

\[
\max_y \ y \cdot 0 \\
S_0^a \leq S_1'y \leq S_0^a \\
y \geq 0
\]

No arbitrage \(\iff\) a risk-neutral pricing measure separates bid and ask prices
Single-period: main results

- No arbitrages $\iff$ There is a risk-neutral measure
- No Arb. $+$ Completeness $\iff$ There is a unique risk-neutral measure
- If there are no arbitrages
  - $b$ attainable $\iff$ $V_0(b) = B_0E^qb$, $\forall q$
  - $b$ not attainable $\Rightarrow$ $V^-(b) \leq V(b) \leq V^+(b)$
  - $\exists q$ separating bid and ask prices
Discrete time models

- $T$ dates: $t = 0, 1, \ldots, T$
- Probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^T, P)$
- $n$ securities $S^1, S^2, \ldots, S^n$
- $S^1$ is a (discrete-time) stochastic process
- $S_t^i$ is $\mathcal{F}_t$-measurable
- Assume $S_t^1 > 0, \forall t$. $S^1$ is called "numeraire". Define the "discounted process" $Z = S / S^1$. 
A "dynamic portfolio" (or "market strategy") is a stochastic process $\theta$. $\theta_t^i$ is the number of shares of security $S^i$ held between $t$ and $t + 1$. $\theta_t^i$ is $\mathcal{F}_t$-measurable.

The discounted value of the portfolio at time $t$ is $\theta_t \cdot Z_t$.

A portfolio strategy is "self-financing" if

$$\theta_t \cdot Z_{t+1} = \theta_{t+1} \cdot Z_{t+1}$$
Because there is a numeraire, any B-arbitrage can be transformed into an A-arbitrage.

A dynamic portfolio $\theta$ is an arbitrage if

\begin{align*}
\mathbb{E} \theta_T \cdot Z_T &> 0 \\
\theta_0 \cdot Z_0 &= 0 \\
\theta_t \cdot Z_{t+1} &= \theta_{t+1} \cdot Z_{t+1}, \quad t = 0, \ldots, T - 1 \\
\theta_T \cdot Z_T &\geq 0
\end{align*}
The arbitrage problem can be set in many equivalent ways. Let us follow the non-recombinant tree representation (King (2002)).

Denote $\mathcal{N}_t$ the set of states at time $t$. For any state $s \in \mathcal{N}_t$, let $\pi(s) \subset \mathcal{N}_{t-1}$ be the parent of $s$ and let $\mathcal{C}(s) \subset \mathcal{N}_{t+1}$ be the set of children of $s$. 
Non-recombinant tree
Example 3

```
<table>
<thead>
<tr>
<th>t=0</th>
<th>t=1</th>
<th>t=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>7.5</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>
```
The Arbitrage problem

To find arbitrages we solve

$$\max_{\theta} \sum_{s \in \mathcal{N}_T} p_s Z_s \cdot \theta_s$$

$$Z_0 \cdot \theta_0 = 0 \quad : y_0$$

$$Z_s \cdot [\theta_s - \theta_{\pi(s)}] = 0 \quad (s \in \mathcal{N}_t, t \geq 1) \quad : y_s$$

$$Z_s \cdot \theta_s \geq 0 \quad (s \in \mathcal{N}_T) \quad : w_s$$
Lagrangian

is

\[
L(\theta; y, w) = \sum_{s \in \mathcal{N}_T} p_s Z_s \cdot \theta_s - \sum_{t=0}^{T} \sum_{s \in \mathcal{N}_t} y_s Z_s \cdot [\theta_s - \theta_a(s)]
\]

\[
- \sum_{s \in \mathcal{N}_T} w_s Z_s \cdot \theta_s, \quad (w_s \leq 0)
\]

\[
= \sum_{s \in \mathcal{N}_T} [p_s - w_s - y_s] Z_s \cdot \theta_s - \\
\sum_{t=0}^{T-1} \sum_{s \in \mathcal{N}_t} [y_s Z_s - \sum_{m \in \mathcal{C}(s)} y_m Z_m] \cdot \theta_s
\]

\[
(w_s \leq 0)
\]
Dual problem

- From the Lagrangian follows the Dual problem

\[
\begin{align*}
  w_s &\leq 0 \quad (s \in \mathcal{N}_T) \\
  (p_s - w_s - y_s)Z_s &= 0 \quad (s \in \mathcal{N}_T) \\
  y_sZ_s - \sum_{m \in \mathcal{C}(s)} y_mZ_m &= (s \in \mathcal{N}_t, t \leq T - 1)
\end{align*}
\]

- There are no arbitrages if and only if the Dual is feasible.

- No arbitrages \iff \exists q \sim p \text{ s.t. } Z_{t-1} = \mathbb{E}^q[Z_t|\mathcal{N}_{t-1}]

- The risk neutral measure \( q \) does not depend on \( p \) (except on sets of measure zero).
Discrete-time: main results

- Same results as in the single-period case
- Fundamental Theorem of Arbitrage:
  No arbitrage $\Leftrightarrow$ There is an equivalent martingale measure
Pricing Contingent Claims: The Buyer’s Problem

Let $\mathbf{b}$ denote a stochastic cash flow and $\tilde{\mathbf{b}}$ its scaled (discounted) version with respect to the numéraire.

Let $V^-(\tilde{\mathbf{b}})$ denote the optimal value of the problem

\[
\max -Z_0 \cdot \theta_0 \\
\text{s.t. } Z_s \cdot (\theta_s - \theta_{\pi(s)}) = \tilde{\mathbf{b}}, \forall s \in \mathcal{N}_t, 1 \leq t \leq T \\
\quad Z_s \cdot \theta_s \geq 0, \forall s \in \mathcal{N}_T
\]

By duality we have $V^-(\tilde{\mathbf{b}}) = \min_{q \in \mathcal{Q}} \mathbb{E}^q \tilde{\mathbf{b}}$

You have to scale back to get $V^-(\mathbf{b})$.\]
Let \( V^+(\tilde{b}) \) denote the optimal value of the problem

\[
\begin{align*}
\min & \quad Z_0 \cdot \theta_0 \\
s.t. & \quad Z_s \cdot (\theta_s - \theta_{\pi(s)}) = -\tilde{b}_s, \quad \forall \ s \in \mathcal{N}_t, 1 \leq t \leq T \\
& \quad Z_s \cdot \theta_s \geq 0, \quad \forall \ s \in \mathcal{N}_T
\end{align*}
\]

By duality we have

\[
V^+(\tilde{b}) = \max_{q \in \mathcal{Q}} \mathbb{E}^q \tilde{b}
\]

You have to scale back to get \( V^+(b) \).
For Further Reading I

What are American Claims?

- An **American Contingent Claim (ACC)** $F$ is a financial instrument generating a real-valued stochastic (cash-flow) process $(F_t)_{t=0,...,T}$.
- At any stage $t = 0, \ldots, T$, the holder of an ACC may decide to take $F_t$ in cash and terminate the process.
- Using this definition, an **American call option** on a stock $S$ with strike price $K$ corresponds to $F = S - K$. 

Variants

- **American put** is obtained by reversing the sign of $F$.
- We can define a **European call option** with maturity $T$ by setting $F_t = 0$ for $t \neq T$.
- **Bermudan call options** having exercise date set $G \subset \{1, \ldots, T\}$ can be defined by setting $F_t = 0$ for $t \notin G$.
- **Russian, look-back, and basket options** can also be accommodated.
When markets are complete

- This means: all stochastic future cash flows can be replicated by existing instruments in the financial markets.
- Pricing strategy: Find a perfectly replicating portfolio, its price today is the price of the future cash flow.
- Otherwise, there would be an arbitrage opportunity.
- But, are markets always complete?
- No, almost never complete!
What characterizes incomplete markets?

Loosely speaking, the sources of uncertainty outnumber the existing instruments.

In incomplete markets, one may not exactly replicate a stochastic cash flow.

Research concentrates on defining and characterizing the range of contingent claim prices consistent with the absence of arbitrage.

This range is determined by the upper hedging and the lower hedging prices, also known as the superreplication and subreplication bounds.
In the absence of arbitrage, the upper hedging price is the value of the least costly self-financing portfolio strategy composed of market instruments whose pay-off is at least as large as the contingent claim pay-off.

This price can also be interpreted from the perspective of a writer (seller) of the contingent claim as the smallest initial wealth required to replicate the contingent claim pay-off at expiration in a self-financed manner.
Lower Hedging Price (Buyer’s Price)

- The **lower hedging price** is the value of the most precious self-financing portfolio strategy composed of market instruments whose pay-off is dominated by the contingent claim pay-off at expiration.

- The lower hedging price can also be interpreted as the largest amount the contingent claim buyer can borrow (in the form of cash or by short-selling stocks) to acquire the claim while paying off his/her debt in a self-financed manner using the contingent claim pay-off at expiration.
Computation of Hedging Prices for European Options by Duality

- Can only be exercised at expiration.
- The upper price is supremum of the expectation of the discounted contingent claim pay-off (at expiration) over all probability measures that make the underlying stock price a martingale.
- The lower hedging price is infimum of the expectation of the discounted contingent claim pay-off (at expiration) over all probability measures that make the underlying stock price a martingale.
- Through convex duality (as we have seen in Part 1).
Computation of Hedging Prices for American Contingent Claims

- Can be exercised any time until expiration.
- The possibility of early exercise complicates pricing formula from the perspective of the buyer.
- Why? Because the holder (buyer) has to choose the optimal exercise time while using the replication argument.
- So, what happens to the pricing formula?
Upper Hedging Prices for American Contingent Claims

- One has to take supremum over all stopping times which represent potential exercise strategies of the contingent claim buyer.
- The upper hedging price is the supremum of the expectation of the discounted contingent claim pay-off (at some time between now and expiration) over all stopping times and all probability measures that make the underlying stock price process a martingale.
- In the discrete time, finite probability space setting, it is a linear programming problem.
- But, what about the lower hedging price?
It is the supremum over all stopping times of the infimum of the expected discounted contingent claim pay-off (at some time between now and expiration) over all probability measures that make the underlying stock price process a martingale.

More precisely, the lower hedging price of an American contingent claim is given by

$$\max_{\tau \in \mathcal{T}} \min_{Q \in \mathcal{Q}} \mathbb{E}^Q[F_\tau]$$

where $\mathcal{T}$ is the set of stopping times, and $\mathcal{Q}$ is the set of all martingale measures and $F_\tau$ is the discounted contingent claim pay-off at time $\tau$; Theorem 12.4 of Chalasani and Jha.
Some Interesting Properties

- The outer maximization over the set $\mathcal{T}$ of stopping times can be replaced by maximization over a set of randomized stopping times.
- We will see that stopping times are in one to one correspondence with binary variable representation of exercise strategies
- while randomized stopping times correspond to their linear relaxations
An Exact Relaxation

- The above observation was made by Pennanen and King (2006), but also earlier by Chalasani and Jha (2001) (no proof given).
- Pennanen and King posed the problem in the hedging space, but there is a problem in their main argument.
- Using their formulation, we prove that we can replace stopping times with randomized stopping times.
- This means: we can price ACCs by linear programming (in discrete time, finite probability spaces)
- We will sketch the proof of this result.
A Consequence of the Result

- Computation of Buyer’s price for ACCs has polynomial complexity.
- Use a basic min-max theorem to exchange max and min.
- The lower hedging price of an American contingent claim is also given by
  \[ \min_{Q \in \mathcal{Q}} \max_{\tau \in \mathcal{T}} \mathbb{E}^Q [F_\tau] \]
- An ACC is almost a scaled-down ECC!
- An ECC (European Contingent Claim) is a security that gives its owner a stochastic cash flow process (adapted to the filtration).
- An ECC may have negative pay-off, e.g., futures contracts, fixed income instruments etc.
More Details on the Stochastic Scenario Tree

- Assume that security prices and other payments are discrete random variables supported on a finite probability space \((\Omega, \mathcal{F}, P)\) whose atoms are sequences of real-valued vectors (asset values) over the discrete time periods \(t = 0, 1, \ldots, T\).

- The market evolves as a discrete, non-recombinant tree (suitable for incomplete markets) where the partition of atoms \(\omega \in \Omega\) generated by matching path histories up to time \(t\) corresponds one to one with nodes \(n \in \mathcal{N}_t\) at level \(t\) in the tree.

- The set \(\mathcal{N}_0\) consists of the root node \(n = 0\), and the leaf nodes \(n \in \mathcal{N}_T\) correspond one to one with the atoms \(\omega \in \Omega\).
In the scenario tree, every node \( n \in \mathcal{N}_t \) for \( t = 1, \ldots, T \) has a unique parent denoted \( \pi(n) \in \mathcal{N}_{t-1} \).

Every node \( n \in \mathcal{N}_t, \ t = 0, 1, \ldots, T - 1 \) has a non-empty set of child nodes \( \mathcal{C}(n) \subset \mathcal{N}_{t+1} \).

The set of all nodes in the tree is denoted by \( \mathcal{N} \).

The set \( \mathcal{A}(n) \) denotes the collection of ascendant nodes or path history of node \( n \) including itself.

\( \mathcal{D}(n) \), the set of descendant nodes of \( n \), again including itself.

The probability distribution \( P \) is obtained by attaching positive weights \( p_n \) to each leaf node \( n \in \mathcal{N}_T \) so that \( \sum_{n \in \mathcal{N}_T} p_n = 1 \).
The Stochastic Scenario Tree (cont’d)

- The price process \( \{S_t\} \) is *adapted* to \( (\mathcal{F}_t)_{t=0}^T \), meaning for all \( t \in \{0, 1, \ldots, T\} \) \( S_t \) depends on which element (nodes) of \( \mathcal{F}_t \) has been realized at \( t \).

- The expected value of process \( S_t \) is uniquely defined by the sum
  \[
  \mathbb{E}^P[S_t] := \sum_{n \in \mathcal{N}_t} p_n S_n.
  \]

- The conditional expectation of \( S_{t+1} \) on \( \mathcal{N}_t \) is given by the expression
  \[
  \mathbb{E}^P[S_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{C}(n)} \frac{p_m}{p_n} S_m.
  \]
The Market

- The market consists of $J + 1$ traded securities indexed by $j = 0, 1, \ldots, J$.
- The prices at node $n$ given by the vector $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$.
- Assume that the security indexed by 0 has strictly positive prices at each node of the scenario tree (standard practice).
- The number of shares of security $j$ held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted $\theta_n^j$.
- Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$ (held for $(t, t + 1]$).
- The value of the portfolio at state $n$ is $S_n \cdot \theta_n = \sum_{j=0}^{J} S_n^j \theta_n^j$. 
Example of Scenario Tree
In our context, a **stopping time** $\tau$ is a random variable that maps each path $\omega \in \Omega$ to a number in $\{0, 1, \ldots, T\}$ with the restriction that for and $t \in \{0, 1, \ldots, T\}$, the indicator random variable $1_{\tau=t}$ is $\mathcal{F}_t$-measurable.

In other words, if there is some path $\omega$ with $\tau(\omega) = t$, and $\omega_t = u$, then for every path $\omega'$ containing $u$, we must have $\tau(\omega') = t$.

For any adapted process $F_t$, we denote by $F_\tau$ the r.v. that maps a path $\omega \in \Omega$ to $F_{\tau(\omega)}(\omega)$. 
Randomized Stopping Times

- A randomized stopping time is a non-negative adapted process (in our case, node function) $X$ with the property that on every path $\omega$ one has $\sum_{t=0}^{T} X_t(\omega) = 1$.

- That is, the sum of random variables $X_0, X_1, \ldots, X_T$ is equal to 1 on every path.

- When a randomized stopping time $X$ is used to describe an exercise strategy, we can think of the value $X_n$ at node $n$ as the probability of exercise at node $n$ given that node $n$ has been reached.
A stopping time $\tau$ corresponds to the randomized stopping time $X^{\tau}$ whose values are restricted to lie in the set $\{0, 1\}$ and defined as follows for any $\omega \in \Omega$, and $t \in \{0, 1, \ldots, T\}$:

$$X^{\tau}(\omega_t) = \begin{cases} 1 & \text{if } \tau(\omega) = t, \\ 0 & \text{otherwise} \end{cases}$$

The ordinary (or pure) stopping times are extreme points of the convex set of randomized stopping times ($= \text{convex hull of stopping times}$); see Baxter and Chacon 1977, *Wahrscheinlichkeitstheorie und verwandte Gebiete*. 
A Literary Example of Appreciation for the Concept of Stopping Time

William Shakespeare, **Julius Caesar**, Act 4, Scene 3 (1599):  
BRUTUS  
*Under your pardon. You must note beside, That we have tried the utmost of our friends, Our legions are brim-full, our cause is ripe: The enemy increaseth every day; We, at the height, are ready to decline. There is a tide in the affairs of men, Which, taken at the flood, leads on to fortune; Omitted, all the voyage of their life Is bound in shallows and in miseries. On such a full sea are we now afloat; And we must take the current when it serves, Or lose our ventures.*
Important Observation

- The set $\mathcal{T}$ of stopping times is in one-to-one correspondence with

$$E = \{ e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted, } \sum_{t=0}^{T} e_t \leq 1 \text{ and } e_t \in \{0, 1\} \text{ P-a.s.} \},$$

- whereas the set by randomized stopping times corresponds to

$$\tilde{E} = \{ e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted, } \sum_{t=0}^{T} e_t \leq 1 \text{ and } e_t \geq 0 \text{ P-a.s.} \}.$$
Important Observation II

- The set of randomized stopping times in finite-state setting
  \[ \tilde{E} = \{ e \mid \sum_{m \in A(n)} e_n \leq 1 \text{ and } e_n \geq 0 \ \forall n \in \mathcal{N} \} \]

  is an integral polytope! (All extreme points are binary-valued).
- The matrix is an interval matrix, and thus is totally unimodular.
- A curious connection to perfect graphs
Theorem (Fulkerson 1973, Chvatal 1975)

The set

\[ P(A) = \{ x : Ax \leq 1, x \geq 0 \} \]

is an integral polytope if and only if

(a) \( A \) is an augmented clique matrix of its derived graph;

(b) the derived graph of \( A \) is perfect.
A Curious Detour to Perfect Graphs II

- For definitions and details see “On Randomized Stopping Points and Perfect Graphs”, Dalang, Trotter and de Werra, *Journal of Combinatorial Theory*.

- **Divertimento**: Can you check these two properties for

\[ \tilde{E} = \{ e \mid \sum_{m \in \mathcal{A}(n)} e_n \leq 1 \text{ and } e_n \geq 0 \ \forall \ n \in \mathcal{N} \} \]
The Buyer’s Problem

Pennanen and King (2006)

\[
\begin{align*}
\text{max} \quad V \\
\text{s.t.} \quad S_0 \cdot \theta_0 &= F_0 e_0 - V \\
S_n \cdot (\theta_n - \theta_{\pi(n)}) &= F_n e_n, \quad \forall \ n \in \mathcal{N}_t, 1 \leq t \leq T \\
S_n \cdot \theta_n &\geq 0, \quad \forall \ n \in \mathcal{N}_T \\
\sum_{m \in A(n)} e_m &\leq 1, \quad \forall \ n \in \mathcal{N}_T \\
\quad e_n &\in \{0, 1\}, \quad \forall \ n \in \mathcal{N}.
\end{align*}
\]
Meaning of the Buyer’s Problem

- The optimal value of $V$ is the largest amount that a potential buyer is willing to disburse for acquiring a given American contingent claim $F$.
- The computation of this quantity via the above integer programming problem is carried out by construction of a least costly (adapted) portfolio process replicating the proceeds from the contingent claim using the market-traded securities so as to avoid any terminal losses.
- The integer variables and related constraints represent the one-time exercise of the American contingent claim.
LP Relaxation of the Buyer’s Problem

A linear programming relaxation of AP1 is the following problem AP2:

\[
\begin{align*}
\text{max} & \quad V \\
\text{s.t.} & \quad S_0 \cdot \theta_0 = F_0 e_0 - V \\
& \quad S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \quad \forall \ n \in \mathcal{N}_t, 1 \leq t \leq T \\
& \quad S_n \cdot \theta_n \geq 0, \quad \forall \ n \in \mathcal{N}_T \\
& \quad \sum_{m \in A(n)} e_m \leq 1, \quad \forall \ n \in \mathcal{N}_T \\
& \quad e_n \geq 0, \quad \forall \ n \in \mathcal{N}.
\end{align*}
\]
An Interesting Result

Theorem

1. There exists an optimal solution to AP2 with $e_n \in \{0, 1\}, \forall \ n \in \mathbb{N}$.
2. The optimal value of AP2 is equal to the optimal value of AP1.
Plan of Proof:

- Assume a fractional optimal solution, i.e., partial exercise of the ACC.
- If partial exercise occurs at node 0, show using LP duality that there exists another optimal solution where full exercise is optimal at node 0.
- If partial exercise occurs at nodes other than the root node, construct another optimal solution with all exercise variables taking on integer values using LP duality.
Case I:

- Assume $e^*$ has $e_0^* \not\in \{0, 1\}$ (partial exercise at root).
- The dual to AP2 can be cast (after some simplifications)

$$
\begin{align*}
\min & \quad \sum_{n \in \mathcal{N}_T} z_n \\
\text{s.t.} & \quad y_0 = 1 \\
& \quad \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \quad \forall \ n \in \mathcal{N} \setminus \mathcal{N}_T \\
& \quad y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \quad \forall \ n \in \mathcal{N} \\
& \quad y_n, z_n \geq 0, \quad \forall \ n \in \mathcal{N}_T.
\end{align*}
$$
Case I (cont’d):

- Complementary slackness implies that third constraint of the above program is satisfied as an equality for the corresponding optimal solution of the dual problem (i.e., $y_0 F_0 - \sum_{m \in N_T} z_m = 0$).
- Since $y_0 = 1$, we have $F_0 = \sum_{m \in N_T} z_m$.
- Thus, the optimal solution to the dual problem is found to be $F_0$.
- Then, by strong duality $F_0$ is the optimal value of AP2.
- A feasible solution to AP2 is $e_0 = 1$, $V = F_0$ and all other variables as zeros with obj. value $F_0$. This is optimal with $e_n \in \{0, 1\}$, $\forall \; n \in N$, thus the proof of case I is complete.
Case II:

- Now assume that optimal solution $e^*$ is such that $e^*_0 = 0$ and $e^*_n \notin \{0, 1\}$ for some $n \in \mathcal{N}$.
- Here, the proof is longer and relies on the following idea.
- Concentrate on a portion of the tree where partial exercise occurs.
- Find the first node in that portion where partial exercise is observed. Let that node be labeled $w$.
- **Claim:** One can always find an optimal solution to AP2 with $e_w \in \{0, 1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w) \setminus \{w\}$.
- Repeating the above procedure and claim, “clean up” the entire tree from fractional values.
Details of Case II:

- Let $I = \{ i | e_i^* \notin \{0, 1 \}, i \in \mathcal{N} \}$.
- Let $G = \{ g | g \in I, A(g) \cap I = \{ g \} \}$.
- Let $w$ be the element with the smallest time index (that is closest to the root) in $G$.
- Note that $e_n^* = 0, \forall n \in A(w) \setminus \{ w \}$ in this case.
- Let $k$ denote the time index for node $w$. 
Details of Case II (cont’d):

- **Claim**: One can always find an optimal solution to AP2 with \( e_w \in \{0, 1\} \) and \( e_i = 0 \) for all \( i \in A(w) \setminus \{w\} \).

- To prove the claim we will consider the two following linear programs to which we will refer as AR1 and AR2 respectively:

\[
\begin{align*}
\max & \quad e_w \\
\text{s.t.} & \quad S_w \cdot (\theta_w - \theta_{\pi(w)}^*) = F_w e_w \\
& \quad S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \quad \forall \ n \in D(w) \setminus \{w\} \\
& \quad S_n \cdot \theta_n \geq 0, \quad \forall \ n \in N_T \cap D(w) \\
& \quad \sum_{m \in A(n) \cap D(w)} e_m \leq 1, \quad \forall \ n \in N_T \cap D(w) \\
& \quad e_n \geq 0, \quad \forall \ n \in D(w),
\end{align*}
\]
Details of Case II (cont’d):

\[
\begin{align*}
\min & \quad e_w \\
\text{s.t.} & \quad S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\
& \quad S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \quad \forall \ n \in \mathcal{D}(w) \setminus \{w\} \\
& \quad S_n \cdot \theta_n \geq 0, \quad \forall \ n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \quad \sum_{m \in A(n) \cap \mathcal{D}(w)} e_m \leq 1, \quad \forall \ n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \quad e_n \geq 0, \quad \forall \ n \in \mathcal{D}(w).
\end{align*}
\]
Details of Case II (cont’d):

- Call an optimal point of AR1 $\bar{\theta}_D(w)$, $\bar{e}_D(w)$ and of AR2 $\tilde{\theta}_D(w)$, $\tilde{e}_D(w)$.
- If the optimal value of AR1 is 1, then we see that $(\bar{\theta}_D(w), \theta^*_{N\setminus D(w)}), (\tilde{e}_D(w), e^*_{N\setminus D(w)})$ form another optimal solution of AP2 with $e_w = 1$.
- For this optimal solution we have $e_w = 1$ and $e_i = 0, \forall i \in A(w) \setminus \{w\}$ (we have also $e_i = 0$, for all $i \in D(w) \setminus \{w\}$ for this solution).
Similarly, if the optimal value of AR2 is 0, then 
\((\tilde{\theta}_D(w), \theta^*_{\mathcal{N}\setminus D(w)}), (\tilde{e}_D(w), e^*_{\mathcal{N}\setminus D(w)})\) form another optimal solution of AP2 with \(e_w = 0\).

Then, for this optimal solution we have \(e_i = 0\), for all \(i \in \mathcal{A}(w)\).
Details of Case II (cont’d):

- Our claim will be proved if we can show that AR2 having an optimal value greater than 0 implies that optimal value of AR1 is 1.
- This is accomplished via LP duality on AR1 and AR2.
- Details are skipped.
Main Argument of PK:

- **Claim** For an optimal solution of AP2 if the contingent claim is exercised partially at a node, then there is another optimal solution in which the contingent claim is fully exercised at that node.
- Unfortunately, this is not always true.
- Let us look at a counterexample.
Previous Example

- Solve the LP relax with CPLEX 9.
- The optimal solution is 2, but fractional ($e_1 = 0.625$).
If the PK proof were correct we should have another optimal solution to this problem with $e_1 = 1$.

However when we add the constraint $e_1 = 1$ and solve the same problem again, we see that the optimal solution becomes 1.8.

This is contradicting the argument in PK.

But, it is possible that the PK claim holds from some ACCs, e.g., American call options.

Alas, we do not yet have a proof, nor a counterexample.
A Consequence of the Theorem

Theorem (PK’06)
If there is no arbitrage in the market price process, the buyer’s price for American contingent claim $F$ can be expressed as

$$\max_{\tau \in T} \min_{Q \in \tilde{Q}} \mathbb{E}^Q[F_{\tau}] = \min_{Q \in \tilde{Q}} \max_{\tau \in T} \mathbb{E}^q[F_{\tau}].$$

Proof is based on the fact that on the LHS max-min is in fact the Buyer’s problem AP1, and some more observations that we give below.
Recall: the set $\mathcal{T}$ of stopping times is in one-to-one correspondence with

$$E = \{ e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted, } \sum_{t=0}^T e_t \leq 1 \text{ and } e_t \in \{0, 1\} \text{ } P\text{-a.s.}\},$$

and can be replaced by randomized stopping times

$$\tilde{E} = \{ e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted, } \sum_{t=0}^T e_t \leq 1 \text{ and } e_t \geq 0 \text{ } P\text{-a.s.}\}.$$

A convenient version of the min-max theorem completes the argument.
The Buyer’s Problem and Gain to Loss Ratio

\[
\begin{align*}
\text{max} & \quad V \\
\text{s.t.} & \quad S_0 \cdot \theta_0 = F_0 e_0 - V \\
& \quad S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \quad \forall \; n \in \mathcal{N}_t, 1 \leq t \leq T \\
& \quad S_n \cdot \theta_n = x_n^+ - x_n^-, \quad \forall \; n \in \mathcal{N}_T \\
& \quad \sum_{m \in \mathcal{A}(n)} e_m \leq 1, \quad \forall \; n \in \mathcal{N}_T \\
& \quad e_n \in \{0, 1\}, \quad \forall \; n \in \mathcal{N} \\
& \quad \sum_{n \in \mathcal{N}_T} p_n x_n^+ \geq \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^-
\end{align*}
\]
Meaning of the Buyer’s Problem in the Gain to Loss Ratio

- The optimal value is the largest amount that a potential buyer is willing to pay for an American contingent claim $F$ under a gain to loss ratio risk limit.

- Strategy: Construct of a least costly portfolio process replicating the proceeds from the ACC by self-financing transactions using traded securities to allow a limited risk in terminal losses.

- Enforce the risk limit that positive terminal positions outweigh negative terminal positions in expectation by a factor $\lambda \geq 1$.

- Good-deal bounds are usually tighter than no-arbitrage bounds. But, the main result fails to hold.
Another Version of Good Deal Bounds

- Use a Sharpe-ratio bound to limit risk exposure
- Gives rise to Mixed Integer Second-Order Cone Optimization problems
- For ECC, large conic programs; see MCP, *Automatica*, 2008.
- For gain-loss ratio good deal bounds on ECCs in single period but infinite state markets, see MCP, *ESAIM CoCV*, 2009.
Extensions

- Our proof of the main result remains valid for dividend-paying assets.
- The result does not hold in the case of proportional transaction costs.
- Chalasani and Jha (2001) claim that stopping times can be replaced by randomized stopping times in the case of proportional transaction costs, but give no proof.
- Can we characterize/classify those cases where the result holds in the gain to loss ratio good deal case?
- Quantile Hedging for American contingent claim buyer (coming soon)...

References for American Claims in Discrete Time


Conclusions

- Optimization Theory is a fundamental tool in Mathematical Finance
- There are interesting problems not only in the classical "Portfolio Selection Theory", but also in "Pricing and Hedging Derivatives Securities"
- This was just a partial introduction. Other interesting problems for optimizers can be found in topics like “Good-deal pricing”, “Efficient Hedging”, ”Model Calibration”, ”Extracting Information from Option Prices”, ”Robust Hedging and Pricing”, etc..
- All these in the next conference!
Giuseppe-Luigi Lagrange:

Si j’avais été riche, je n’aurais jamais consacré ma vie aux mathématiques!